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Superspace Formulation of $N = 4$ Super Yang-Mills Theory with a Central Charge

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(Master's Thesis)*

Abstract

A superspace formulation using superconnections and supercurvatures is specifically constructed for $N = 4$ extended super Yang-Mills theory with a central charge in four dimensions, first proposed by Sohnius, Stelle and West long ago. We find that the constraints, almost uniquely derived from the possible spin structure of the multiplet, can be algebraically solved which results in an off-shell supersymmetric formulation of the theory on the superspace.

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1 Introduction

Theory of elementary particle is widely studied for the construction of a unified theory which could explain every simple aspect of nature including matters and interactions. We know that such a theory can be realized to some extent as a quantum field theory (QFT) based on the gauge principle. In fact, so called the standard theory of particle physics has been constructed as a quantum field theory with gauge invariance (Yang-Mills theory). There are many successful

examples where description of nature in the standard theory agrees with experimental data with a considerably surprising accuracy.

Though phenomenologically successful, the standard theory can not be interpreted as a sufficient unified theory of particle physics. This is clearly seen by noting the following fact. A consistent and unified scheme which can treat the two fundamental statistics of particles, i.e. bosonic and fermionic statistics, should be essentially implemented into a unified theory. In fact, fundamental matters in particle physics—quarks and leptons—are all described as fermions, while fundamental interactions—photons (which mediate electromagnetic interaction), W and Z bosons (weak), gluons (strong) and gravitons (gravity)—are all described as bosons. So naively we expect that any theory which unifies these fundamental matters and interactions should naturally treat bosons and fermions in a unified way. The standard theory dose not contain such a unified scheme.

Supersymmetry (SUSY) serves one systematic, somewhat realistic and almost unique possibility which treats bosons and fermions as unified objects consistent with relativistic spacetime structure. In a supersymmetric theory, bosons and fermions are always introduced as a pair, and forms a multiplet over which several kinds of transformations, called supertransformations, are defined. In this sense, the bosons and fermions are treated in a unified manner and are called superpartners. The theory is said to be supersymmetric when constructed to be symmetric under the supertransformations of the superpartners. Since a supertransformation converts a boson into a fermion, or vice versa, it has to be generated by a fermionic charge, called a supercharge. Such transformations, or charges, which must be consistent with relativistic symmetry, namely the Poincaré symmetry, are uniquely prescribed by a graded Lie algebra, which is called the supersymmetry algebra (Coleman-Mandula no-go theorem, Haag-Łopuszanski-Sohnius theorem).

Phenomenologically, only $N = 1$ supersymmetry[1], in which a set of supercharges is conserved, may seem to be realistic. In fact, the minimal supersymmetric standard model (MSSM) with $N = 1$ supersymmetry can be interpreted as a successful phenomenological model from, for instance, the following reasons. Firstly, MSSM unifies the coupling constants of the gauge group $SU(3) \times SU(2) \times U(1)$ in the model as the meeting of the renormalization group flows of the each coupling constant at the GUT scale ($\sim 10^{16}$ GeV), implying that the gauge group itself is unified in the scale. This unification of the coupling constants dose not occur in the usual standard model without $N = 1$ supersymmetry. Secondly, so called the problem of naturalness, or the problem of hierarchy of mass, can be solved in MSSM by the cancellation of some bad divergences of self-energies due to the high degrees of freedom of supersymmetry.

From the viewpoint of the unification, however, so called extended supersymmetry, or $N \geqslant 2$ supersymmetry, where several sets of supercharges are conserved, may give more natural schema in many cases, as in the following examples. First, those extended degrees of supersymmetry in four dimensions can be naturally interpreted as dimensionally reduced degrees of freedom of some higher dimensional supersymmetric theories, including superstring theories and supergravity theories in ten or eleven dimensions. Such kind of naive correspondence of extended supersymmetry and higher dimensional supersymmetry may play more explicitly a role in some cases as in the AdS/CFT correspondence, where the type IIA superstring (or supergravity) on $\mathrm{AdS}_5 \times S^5$ is naturally related to $N = 4$ $U(N)$ super Yang-Mills (SYM) theory in four dimensions as a low energy effective theory. Second, in the context of duality, which has been one of key notions in recent theoretical works, extended supersymmetry naturally appears and gives many fruitful topics, both from physical and mathematical point of view. Third, and as the fact which gives most directly the motivation of this article, supersymmetry on a lattice in four dimensions should be naturally and intrinsically interpreted as (twisted) $N = 4$ extended supersymmetry as is proposed in recent works [2, 3]. In that paper, so called twisted supersymmetry, an exotic version of extended supersymmetry which may be related with some topological theories or

[1]In the following, we denote the number of sets of conserved supercharges by N.

some BRST quantized gauge theories [4, 5, 6, 7, 8, 9], is introduced on a lattice, based on the facts that supercharges conserved in a twisted supersymmetric theory are naturally associated with the simplex structure of the lattice. Introduced on the lattice, internal degrees of freedom of extended supersymmetry may be interpreted as flavor degrees of freedom through the Dirac-Kähler mechanism. Species doublers on the lattice, notorious obstacles which appear by introducing chiral fermions on a lattice, may also be identified these internal or flavor degrees. Thus there arises the possibility that extended $N = 4$ supersymmetry on a lattice can also give a unified solution to the problem of chiral fermions on the lattice.

We should briefly remark here why we need to introduce a lattice theory. Quantum field theory deals with inevitably infinite degrees of freedom of the fields. This infinite degrees of freedom comes from that spacetime in quantum field theory is assumed to be a four-dimensional continuum. Continuous spacetime has infinitely many degrees of freedom in an infinitely small area, or, in its momentum representation, in an infinitely large momentum region over which a divergent integration will be produced. Those divergences are thus essential and unavoidable in quantum field theories, so that they have to be estimated systematically as finite quantities to make a rigorous calculation. Such a technical scheme to evaluate those divergences is called a regularization. Introducing a lattice structure into spacetime is considered to be one of the most natural and important regularizations among others especially by the following reasons. First, theory on a lattice is theoretically important from the viewpoint of the unification. In fact, there is an approach to formulate gravity as well as the other three interactions and matters on a (random) lattice in a unified manner, and we know such a formulation is successful in two dimensions. Though no such theory has been completely formulated in four dimensions, we expect the approach to a unified theory on a four-dimensional random lattice could also be a successful scheme for the unification. Second, a theory on a lattice may be analyzed numerically to make a realistic and practical computation. As is well known this is the case in lattice QCD, which shows the prominence of the lattice regularization especially in a non-perturbative region where quantitative analysis can hardly be done in other approaches.

Thus to construct a supersymmetric theory on a lattice may seem to serve a candidate of a realistic unification including gravity.

We should especially note here that the formulation for introducing supersymmetry on a lattice in the paper [2, 3] inevitably needs the full twisted $N = 4$ off-shell supersymmetry (i.e., supersymmetry respected exactly without using classical solutions) in a continuous four-dimensional spacetime. This aspect should be compared with the fact that in other theories with twisted supersymmetry only a part of supercharges, especially the scalar charges, is taken into account so that off-shell formulation in such theories is merely corresponding to only a part of supercharges, not corresponding to the full supercharges.

One may thus consider to construct an off-shell formulation of a full $N = 4$ supersymmetric, especially super Yang-Mills, theory. However, the $N = 4$ super Yang-Mills theory with the full internal symmetry $SU(4)$ in four dimensions has been formulated only on-shell [10]. Instead, there is one known off-shell formulation of $N = 4$ super Yang-Mills theory with the internal symmetry $USp(4)$ [11, 12]. This model contains essentially a central charge to prohibit higher spin components [13]. Since an off-shell formulation of a supersymmetric theory is constructed most clearly and systematically by superspace formulation, these off-shell $USp(4)$ model should be naturally formulated on a superspace. Here we have to emphasize that still no superspace formulation of $N = 4$ super Yang-Mills theory in four dimensions is known.

Motivated by the facts above, I have attempted in this work to construct a superspace formulation, mentioned briefly in [12], of the $USp(4)$ super Yang-Mills theory in four dimensions. For the task, I applied a formulation using superconnections and supercurvatures with the manifest gauge covariance. I set up constraints on the superfields in the formulation by noting the contents of multiplets as well as their spins in the $USp(4)$ model and solve the constraints to

derive the supertransformations of the contents. There remains some difficulties which should be resolved by the future works.

This article is organized as follows. In section 2 we review some foundations of extended supersymmetry algebra in four dimensions, particularly one with central charges. In section 3 we discuss twisted supersymmetries in four dimensions since those ideas serve an essential background to this article. In section 4 we present the construction of the superspace formulation of the $USp(4)$ model, which includes the main results of this article. Finally we conclude in section 5, and mention some possible ideas to complete our superspace formulation.

2 N-Extended Supersymmetry in Four Dimensions

In this section, we briefly review some general aspects of the N-extended superalgebra in four dimensions and its irreducible representations [14, 15, 16]. Especially we consider supersymmetry with central charges. Notational and technical details are listed in the appendix.

2.1 Superalgebra

N-extended superalgebra (super Poincaré algebra) in four dimensions is prescribed by the following (anti-)commutation relations:

$$\{Q_{i\alpha}, Q_{j\beta}\} = C_{\alpha\beta} Z_{ij}, \qquad \{\overline{Q}^i{}_{\dot\alpha}, \overline{Q}^j{}_{\dot\beta}\} = C_{\dot\alpha\dot\beta}\overline{Z}^{ij}, \tag{2.1}$$

$$\{Q_{i\alpha}, \overline{Q}^j{}_{\dot\beta}\} = 2\delta_i{}^j (\sigma^\mu)_{\alpha\dot\beta} P_\mu, \qquad [P_\mu, P_\nu] = 0, \tag{2.2}$$

$$[Q_{i\alpha}, P_\mu] = 0, \qquad [\overline{Q}^i{}_{\dot\alpha}, P_\mu] = 0, \tag{2.3}$$

$$[J_{\mu\nu}, Q_{i\alpha}] = \frac{i}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_{i\beta}, \qquad [J_{\mu\nu}, \overline{Q}^{i\dot\alpha}] = \frac{i}{2}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\overline{Q}^{i\dot\beta}, \tag{2.4}$$

$$\begin{aligned}[J_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \qquad [J_{\mu\nu}, J_{\rho\sigma}] = i\,(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\nu\sigma}J_{\mu\rho} \\ &\qquad - \eta_{\mu\rho}J_{\nu\sigma} + \eta_{\mu\sigma}J_{\nu\rho}),\end{aligned} \tag{2.5}$$

$$[R^a, Q_{i\alpha}] = (X^a)_i{}^j Q_{j\alpha}, \qquad [R^a, \overline{Q}^i{}_{\dot\alpha}] = -(X^a)_j{}^i \overline{Q}^j{}_{\dot\alpha}, \tag{2.6}$$

$$[R^a, P_\mu] = 0, \qquad [R^a, J_{\mu\nu}] = 0, \tag{2.7}$$

$$[Z_{ij}, \mathrm{any}] = 0, \qquad [\overline{Z}^{ij}, \mathrm{any}] = 0, \tag{2.8}$$

where the supercharges $Q_{i\alpha}$, $\overline{Q}^i{}_{\dot\alpha}$, related by Hermitian conjugation as

$$\overline{Q}^i{}_{\dot\alpha} = (Q_{i\alpha})^\dagger, \tag{2.9}$$

are represented as Weyl spinors w.r.t. Lorentz transformation $SO(1,3) \cong SL(2,\mathbb{C})$ in Minkowski spacetime, and as the fundamental $\boldsymbol{N}$ and $\overline{\boldsymbol{N}}$ representations w.r.t. the R-symmetry (internal symmetry)[2] $SU(N)$, i.e.

$$Q_{i\alpha} \in (\mathbf{2}, \mathbf{0}, \boldsymbol{N}), \quad \overline{Q}^i{}_{\dot\alpha} \in (\mathbf{0}, \mathbf{2}, \overline{\boldsymbol{N}}), \qquad \text{under} \quad SL(2,\mathbb{C}) \times SU(N); \tag{2.10}$$

Z_{ij} and $\overline{Z}^{ij}$ are central charges which satisfy the relations[3]

$$Z_{ij} + Z_{ji} = 0, \quad \overline{Z}^{ij} + \overline{Z}^{ji} = 0, \tag{2.11}$$

[2]In the following, we concentrate only on the R-symmetry $SU(N)$ for simplicity. The extension to the case $U(N) = U(1) \times SU(N)$, or to more general cases, is trivial. Note, however, if $N = 4$ we cannot adopt the symmetry $U(4)$ for the super Yang-Mills multiplet in four dimensions because of the CPT theorem.

[3]In order for the central charges to be consistently introduced in the superalgebra, the R-symmetry group should be restricted to some extent as is seen below.

i.e.

$$Z_{ij} \in \frac{\boldsymbol{N(N-1)}}{\mathbf{2}}, \quad \overline{Z}^{ij} \in \overline{\frac{\boldsymbol{N(N-1)}}{\mathbf{2}}} \qquad \text{under} \quad SU(N), \tag{2.12}$$

and

$$\overline{Z}^{ij} = (Z_{ij})^*, \quad \overline{Z} + Z^\dagger = 0; \tag{2.13}$$

$\eta_{\mu\nu}$ is the flat spacetime metric

$$\eta_{\mu\nu} = (+,-,-,-) \quad \text{(Minkowski)}, \tag{2.14}$$

with its contragradient $\eta^{\mu\nu}$ such that

$$\eta^{\mu\rho}\eta_{\rho\nu} = \eta^\mu{}_\nu := \delta^\mu{}_\nu; \tag{2.15}$$

P_μ is the four momentum and $J_{\mu\nu}$ is the Lorentz generator

$$P_\mu \in (\mathbf{2},\mathbf{2}), \quad J_{\mu\nu} \in (\mathbf{3},\mathbf{0}) \oplus (\mathbf{0},\mathbf{3}) \qquad \text{under} \quad SL(2,\mathbb{C}), \tag{2.16}$$

with the convention

$$\exp\left(-\frac{i}{2}\omega^{\mu\nu}J_{\mu\nu}\right) \in SO(1,3); \tag{2.17}$$

R^a is the generator of the R-symmetry (internal symmetry) $SU(N)$, i.e.

$$R^a \in \mathfrak{su}(N) \tag{2.18}$$

with the convention

$$\exp\left(i\sum_a t^a R^a\right) \in SU(N), \tag{2.19}$$

and $(X^a)_i{}^j$ is its adjoint representation

$$(X^a)_i{}^j \in (\boldsymbol{N^2 - 1}) \qquad \text{under} \quad SU(N), \tag{2.20}$$

with

$$(X^{a*})^i{}_j := ((X^a)_i{}^j)^* = (X^a)_j{}^i \quad \text{(Hermitian)}; \tag{2.21}$$

$C_{\alpha\beta}$ and $C_{\dot{\alpha}\dot{\beta}}$ are the $SL(2,\mathbb{C})$ invariant tensors with relations

$$C_{\alpha\beta} + C_{\beta\alpha} = 0, \quad C_{\dot{\alpha}\dot{\beta}} + C_{\dot{\beta}\dot{\alpha}} = 0, \tag{2.22}$$

i.e.

$$C_{\alpha\beta} \in (\mathbf{1},\mathbf{0}), \quad C_{\dot{\alpha}\dot{\beta}} \in (\mathbf{0},\mathbf{1}) \qquad \text{under} \quad SL(2,\mathbb{C}), \tag{2.23}$$

and

$$C_{\dot{\alpha}\dot{\beta}} = (C_{\alpha\beta})^*, \tag{2.24}$$

while also $C^{\alpha\beta}$ and $C^{\dot{\alpha}\dot{\beta}}$ are defined as

$$C_{\alpha\gamma}C^{\beta\gamma} = \delta_\alpha{}^\beta, \quad C_{\dot{\alpha}\dot{\gamma}}C^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}{}^{\dot{\beta}}; \tag{2.25}$$

$(\sigma^\mu)_{\alpha\dot{\beta}}$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}$ are defined by the Pauli matrices τ^i as

$$\sigma^\mu = (\mathbf{1}, \tau^i), \quad \bar{\sigma}^\mu = (\mathbf{1}, -\tau^i) \tag{2.26}$$

in Minkowski spacetime, and interpreted as

$$(\sigma^\mu)_{\alpha\dot{\beta}} \in (\mathbf{2},\mathbf{2}), \quad (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \in (\mathbf{2},\mathbf{2}) \qquad \text{under} \quad SL(2,\mathbb{C}), \tag{2.27}$$

with relations

$$(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta = 2\eta^{\mu\nu}\delta_\alpha{}^\beta, \quad (\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} = 2\eta^{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}}; \tag{2.28}$$

and $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are defined by

$$(\sigma^{\mu\nu})_\alpha{}^\beta := \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta, \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} := \frac{1}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{2.29}$$

which are interpreted as

$$\sigma^{\mu\nu} \in (\mathbf{3},\mathbf{0}), \quad \bar{\sigma}^{\mu\nu} \in (\mathbf{0},\mathbf{3}) \qquad \text{under} \ \ SL(2,\mathbb{C}). \tag{2.30}$$

Let us quickly recall how central charges are consistently implemented into the superalgebra above. For this we consider the case where the algebra has more general internal symmetry which includes the R-symmetry $\mathfrak{su}(N)$. The Coleman-Mandula theorem restricts such an internal symmetry to be generated by Lorentz invariant and compact Lie algebra $\mathcal{A}$. More precisely, we can take as $\mathcal{A} = \mathfrak{su}(N) \oplus \mathcal{A}_1 \oplus \mathcal{A}_2$ with $\mathcal{A}_1$ and $\mathcal{A}_2$ being invariant semi-simple and Abelian subalgebras, respectively. Accordingly let us denote the irreducible representations of this internal symmetry algebra by $Q_{I\alpha}, \ \overline{Q}^J{}_{\dot{\beta}}$ with $I = (i, \mathfrak{i}), \ J = (j, \mathfrak{j})$, and let the Hermitian generators of this algebra be B^l including R^a. Assume these generators satisfy the relations

$$[B^l, Q_{I\alpha}] = (S^l)_I{}^J Q_{J\alpha}, \qquad [B^l, \overline{Q}^I{}_{\dot{\alpha}}] = -(S^l)_J{}^I \overline{Q}^J{}_{\dot{\alpha}}, \tag{2.31}$$

$$[B^l, P_\mu] = 0, \qquad [B^l, J_{\mu\nu}] = 0. \tag{2.32}$$

Since $\{Q_{I\alpha}, Q_{J\beta}\}$ is symmetric under the exchange of $(I,\alpha) \leftrightarrow (J,\beta)$, it should have most generally the form

$$\{Q_{I\alpha}, Q_{J\beta}\} = C_{\alpha\beta}Z_{IJ} + \frac{1}{2}(\sigma^{\mu\nu})_{\alpha\beta}(Y_{\mu\nu})_{IJ}, \tag{2.33}$$

where Z_{IJ} is antisymmetric under $I \leftrightarrow J$ while $(Y_{\mu\nu})_{IJ}$ is symmetric, and $(\sigma^{\mu\nu})_{\alpha\beta} := (\sigma^{\mu\nu})_\alpha{}^\gamma C_{\gamma\beta}$ is symmetric under $\alpha \leftrightarrow \beta$. According to the Coleman-Mandula theorem, we have to set

$$Z_{IJ} = \sum_l (a^l)_{IJ} B^l \in \mathcal{A}, \qquad (Y_{\mu\nu})_{IJ} = Y_{IJ} J_{\mu\nu}, \tag{2.34}$$

Then the Jacobi identity, firstly, w.r.t. $P_\mu, \ Q_{I\alpha}, \ Q_{J\beta}$ leads to that $Y_{IJ} = 0$, secondly, the identity w.r.t. $B^l, \ Q_{I\alpha}, \ Q_{J\beta}$ shows that linear combinations Z_{IJ} generates an invariant subalgebra $\mathcal{Z}$ of $\mathcal{A}$, and, thirdly, the identity w.r.t. $Q_{I\alpha}, \ Q_{J\beta}, \ \overline{Q}^K{}_{\dot{\gamma}}$ and the fact that B^l is Hermitian prove that $\mathcal{Z}$ is Abelian. Thus we find that $\mathcal{Z} = \mathcal{A}_2$ which is the center of the algebra $\mathcal{A}$, namely,

$$[Z_{IJ}, B^l] = 0, \qquad \{Q_{I\alpha}, Q_{J\beta}\} = C_{\alpha\beta}Z_{IJ}. \tag{2.35}$$

Therefore a nonzero central charge exists if and only if the internal symmetry algebra has the invariant Abelian subalgebra, and, of course, one has to work on supersymmetry for $N \geqslant 2$. In what follows, we only consider the R-symmetry and the symmetry generated by the center $\mathcal{Z}$ as nontrivial internal symmetries of our system, namely the case $\mathcal{A} = \mathfrak{su}(N) \oplus \mathcal{Z}$. Since $\mathcal{Z}$ is Abelian, its action is a trivial scalar multiplication so that its adjoint representation is trivial. Thus we can label the quantities simply by indices $I = i, \ J = j$ and treats only R^a as nontrivial generators of the internal symmetry, which in turn leads to the superalgebra listed at the beginning of this section.

2.2 Central Charges

In the preceding section, we have seen central charges can be interpreted as generators of Abelian internal symmetry. However, one can not introduce a nonzero central charge yet consistently to an arbitrary structure of the R-symmetry. In fact, the Jacobi identity w.r.t. R^a, $Q_{i\alpha}$, $Q_{j\beta}$ together with that $[R^a, Z_{ij}] = 0$ leads to

$$(X^a)_i{}^k Z_{kj} + Z_{ik}(X^a)_j{}^k = 0, \tag{2.36}$$

or equivalently, since X^a is Hermitian and $Z_{ij} = \sum_{B^l \in \mathcal{A}} (a^l)_{ij} B^l$,

$$(X^a)_i{}^k (a^l)_{kj} = -(a^l)_{ik}(X^{a*})^k{}_j. \tag{2.37}$$

If $(a^l)_{ij}$ is nondegenerate, i.e. if the matrix $((a^l)_{ij})$ has its inverse, the above condition is more directly written as

$$(X^a)_i{}^j = (a^l)_{ik}(X^{a*})^k{}_l (a^l)^{lj}, \tag{2.38}$$

where $(a^l)^{ij} := (((a^l)^T)^{-1})^{ij}$. These equations show that each coefficient $(a^l)_{ij}$ relates each representation of R-symmetry generator X^a to its conjugate representation X^{a*}. Central charges can exist if and only if such an intertwiner a^l does exist. In other words, the representation of the R-symmetry algebra, or at least some invariant subalgebra, has to be (pseudo) real, namely, the R-symmetry algebra has to be automorphic, for a central charge to exist.

If we persist to require the full $SU(N)$ as the R-symmetry, such intertwiner a^l does not exist except the case[4] $N = 2$, since for $N \geqslant 3$ the fundamental representation $\boldsymbol{N}$ of $SU(N)$ is not real, i.e. $\boldsymbol{N}$ and $\overline{\boldsymbol{N}}$ are inequivalent[5]. These facts can also be seen easily by a direct computation; let eq. (2.37) hold for all generators X^a $(a = 1, \cdots, N^2 - 1)$ in $\mathfrak{su}(N)$, multiply the both side by $(X^a)_m{}^n$, take summation w.r.t. $a = 1, \cdots, N^2 - 1$ by using the completeness relation eq. (A.59) in $\mathfrak{su}(N)$, and then take contraction w.r.t. indices n and j to give

$$0 = (N^2 - N - 2)(a^l)_{im} = (N - 2)(N + 1)(a^l)_{im}, \tag{2.39}$$

which shows that nonzero intertwiner a^l exists only if $N = 2$.

Thus we have to break the full R-symmetry group $SU(N)$ into some automorphic subgroup of $SU(N)$ in order to introduce a nonzero central charge. If N is even, say $N = 2n$, the unitary symplectic group[6] $USp(2n) \subset SU(2n)$ can be taken as a candidate, for, $USp(2n)$ contains the invariant tensor Ω_{ij} such that

$$(X^a)_i{}^j = \Omega_{ik}(X^{a*})^k{}_l \Omega^{lj}, \quad \Omega^{ij} := ((\Omega^T)^{-1})^{ij}. \tag{2.40}$$

Later we consider the automorphic subgroup $USp(4) \subset SU(4)$ for $N = 4$ super Yang-Mills theory with a central charge, which is the main subject of this article. Another nontrivial example could be served by the spinor representation of $SO(N) \subset SU(N)$ for some appropriate N, where the charge conjugation matrix, or equivalently the B-conjugation matrix[7], can be used as invariant intertwiners.

Central charges can be interpreted as some sort of additional masses. This can be seen by the following observations. Firstly, by diagonalizing the on-shell superalgebra w.r.t. the central charges as will be seen in the next section, we can show the BPS bound relation

$$2m \geqslant z^l, \tag{2.41}$$

[4]For $N = 2$, $SU(2) \cong USp(2)$ contains such an intertwiner; the $SU(2)$ invariant tensor ε_{ij} plays the role.

[5]See appendix A.3.3.

[6]See appendix A.4.

[7]See appendix B.

where m is the mass of the on-shell algebra and z_a are the eigenvalues of the central charges. This relation allows one to interpret z_a, i.e. the central charges, as some kind of masses. Secondly, if we consider a superalgebra in higher dimensions and then take a dimensional reduction by, for instance, the Kaluza-Klein compactification of spacetime, we could obtain a superalgebra in four dimensions with a central charge which originates from the momentum corresponding to the compactified direction. In such case, the central charge can be clearly regarded as a mass of an extra dimension. Thirdly, by an appropriate spontaneous breaking of a gauge symmetry in some extended supersymmetric gauge theories, a central charge emerges as a nonzero vev (of a moduli) both algebraically and field theoretically [13]. In this formulation, origins of some part of central charges are gauge bosons in higher dimensions and as an effect are related somewhat directly to the momenta in the higher dimensions. In fact, as an on-shell relation one can show that $P^\mu P_\mu = \sum_l (z^l)^2$, which is identical to the BPS condition.

2.3 Representations

Let us now examine the irreducible one particle representations of the superalgebra eqs. (2.1)–(2.8).

2.3.1 Massive Multiplet without Central Charge

First we consider the superalgebra with no central charge. For massive multiplets with mass $m > 0$ i.e. $P^2 = m^2$, we can take the rest frame in which $P^\mu = (m, \mathbf{0})$. Then relevant anticommutation relations are written as

$$\{Q_{i\alpha}, \overline{Q}^j{}_{\dot\beta}\} = 2m\delta_{\alpha\dot\beta}\delta_i{}^j, \tag{2.42}$$

$$\{Q_{i\alpha}, Q_{j\beta}\} = 0, \qquad \{\overline{Q}^i{}_{\dot\alpha}, \overline{Q}^j{}_{\dot\beta}\} = 0. \tag{2.43}$$

Then defining

$$a_{i\alpha} := \frac{1}{\sqrt{2m}} Q_{i\alpha}, \quad (a_{i\alpha})^\dagger := \frac{1}{\sqrt{2m}} \overline{Q}^i{}_{\dot\alpha}, \tag{2.44}$$

we find that

$$\{a_{i\alpha}, (a_{j\beta})^\dagger\} = \delta_\alpha{}^\beta \delta_i{}^j, \tag{2.45}$$

$$\{a_{i\alpha}, a_{j\beta}\} = 0, \qquad \{(a_{i\alpha})^\dagger, (a_{j\beta})^\dagger\} = 0. \tag{2.46}$$

Hence we obtain $2N$ sets of fermionic creation-annihilation operators which constructs the irreducible representation with total of $2^{2N}(2j+1)$ states on a Clifford vacuum with spin j. The maximum spin for such representations is $N/2 + j$.

2.3.2 Massless Multiplet without Central Charge

For massless multiplets, we can take a light-cone frame where $P^2 = 0$, so for instance $P^\mu = (P^0, 0, 0, -P^0)$, $P^0 > 0$. Defining

$$a_i := \frac{1}{2\sqrt{P^0}} Q_{1i}, \quad (a_i)^\dagger := \frac{1}{2\sqrt{P^0}} \overline{Q}^i{}_{\dot 1}, \tag{2.47}$$

$$b_i := \frac{1}{2\sqrt{P^0}} Q_{2i}, \quad (b_i)^\dagger := \frac{1}{2\sqrt{P^0}} \overline{Q}^i{}_{\dot 2}, \tag{2.48}$$

we find that

$$\{a_i, (a_j)^\dagger\} = \delta_i{}^j, \tag{2.49}$$

$$\{a_i, a_j\} = 0, \qquad \{(a_i)^\dagger, (a_j)^\dagger\} = 0, \tag{2.50}$$

while

$$\{b_i, (b_j)^\dagger\} = 0, \quad \{b_i, b_j\} = 0, \quad \{(b_i)^\dagger, (b_j)^\dagger\} = 0, \tag{2.51}$$

$$\{a_i, b_j\} = 0, \quad \{(a_i)^\dagger, (b_j)^\dagger\} = 0, \quad \{a_i, (b_j)^\dagger\} = 0, \quad \{(a_i)^\dagger, b_j\} = 0. \tag{2.52}$$

Hence we obtain only N sets of fermionic creation-annihilation operators, with b_i, $(b_i)^\dagger$ merely represented as 0 on a Hilbert representation space. These operators thus create total of 2^N states[8] on a Clifford vacuum with helicity λ. The maximum helicity for such representations is clearly $N/2 + \lambda$.

2.3.3 Multiplet with Nonzero Central Charges

Finally let us consider the multiplets on which central charges take nonzero values. We take a massive rest frame, by the reason which will be clear shortly, with momentum $P^2 = m^2$, $P^\mu = (m, \mathbf{0})$. The relevant anticommutators are

$$\{Q_{i\alpha}, \overline{Q}^j{}_{\dot\beta}\} = 2m\delta_{\alpha\dot\beta}\delta_i{}^j, \tag{2.53}$$

$$\{Q_{i\alpha}, Q_{j\beta}\} = C_{\alpha\beta}Z_{ij}, \qquad \{\overline{Q}^i{}_{\dot\alpha}, \overline{Q}^j{}_{\dot\beta}\} = C_{\dot\alpha\dot\beta}(Z^*)^{ij}. \tag{2.54}$$

Since $[Z_{ij}, \text{any}] = [(Z^*)^{ij}, \text{any}] = 0$, the central charges can be simultaneously diagonalized. Further, because Z_{ij} is antisymmetric, we can take, by a suitable orthogonal (thus unitary) transformation $(U_i{}^j)$ w.r.t. the internal indices, a basis in which the central charges take the standard form as in

$$Z_{ij} = U_i{}^k (Z^{\text{std}})_{kl} U_j{}^l, \tag{2.55}$$

$$(Z^{\text{std}})_{ij} := iC \otimes Z^{\text{d}} \quad (N \text{ even}) \qquad (Z^{\text{std}})_{ij} := \begin{pmatrix} iC \otimes Z^{\text{d}} & 0 \\ 0 & 0 \end{pmatrix} \quad (N \text{ odd}), \tag{2.56}$$

where $Z^{\text{d}} = \text{diag}(z^1, \cdots, z^{\text{rank}(Z)})$ with $z^l \geqslant 0$ being the eigenvalues of (Z_{ij}) and the tensor product $(A_i{}^k) \otimes (B_j{}^l) = (A_i{}^k B_j{}^l)$ is represented by the dictionary order

$$(i, j) = (1, 1), \cdots, (n, 1), \cdots, (1, n), \cdots, (n, n), \tag{2.57}$$

and similarly for (k, l). Then by pursuing the transformation

$$\tilde{Q}_{i\alpha} = (U^{-1})_i{}^j Q_{j\alpha}, \quad (\tilde{Q}_{i\alpha})^\dagger = \tilde{\overline{Q}}{}^i{}_{\dot\alpha} = U_j{}^i \overline{Q}^j{}_{\dot\alpha}, \tag{2.58}$$

and by labeling as $i = (am)$ $(m = 1, \cdots, [N/2])$ just corresponding to the standard form Z^{std}, we find that

$$\{\tilde{Q}_{am\alpha}, (\tilde{Q}_{bn\beta})^\dagger\} = 2m\delta_\alpha{}^\beta \delta_a{}^b \delta_m{}^n, \tag{2.59}$$

$$\{\tilde{Q}_{am\alpha}, \tilde{Q}_{bn\beta}\} = iC_{\alpha\beta}C_{ab}(Z^{\text{d}})_{mn}, \qquad \{(\tilde{Q}_{am\alpha})^\dagger, (\tilde{Q}_{bn\beta})^\dagger\} = iC^{\alpha\beta}C^{ab}(Z^{\text{d}})^{mn}. \tag{2.60}$$

Note here that for the case of N odd we generalize the diagonal matrix Z^{d} as

$$Z^{\text{d}} = \text{diag}(z^1, \cdots, z^{\text{rank}\, Z}, 0, \cdots, 0). \tag{2.61}$$

[8] We work on a CPT invariant field theory so that we require the CPT-conjugate of a multiplet is also included in the whole multiplet. Thus unless the multiplet composed of the 2^N states created purely by the supercharges is CPT-self-conjugate, it needs its CPT-conjugate to be accompanied so that the whole multiplet consists of 2^{N+1} states.

Hence letting

$$a_{m\alpha} := \frac{1}{\sqrt{2}}\left(\tilde{Q}_{1m\alpha} + iC_{\alpha\gamma}(\tilde{Q}_2{}^m{}_\gamma)^\dagger\right), \quad (a_{m\alpha})^\dagger := \frac{1}{\sqrt{2}}\left((\tilde{Q}_{1m\alpha})^\dagger - iC^{\alpha\gamma}\tilde{Q}_2{}^m{}_\gamma\right), \tag{2.62}$$

$$b_{m\alpha} := \frac{1}{\sqrt{2}}\left(\tilde{Q}_{1m\alpha} - iC_{\alpha\gamma}(\tilde{Q}_2{}^m{}_\gamma)^\dagger\right), \quad (b_{m\alpha})^\dagger := \frac{1}{\sqrt{2}}\left((\tilde{Q}_{1m\alpha})^\dagger + iC^{\alpha\gamma}\tilde{Q}_2{}^m{}_\gamma\right), \tag{2.63}$$

we obtain

$$\{a_{m\alpha}, (a_{n\beta})^\dagger\} = \delta_{\alpha\beta}(2m\delta_{mn} + (Z^{\mathrm{d}})_{mn}), \quad \{b_{m\alpha}, (b_{n\beta})^\dagger\} = \delta_{\alpha\beta}(2m\delta_{mn} - (Z^{\mathrm{d}})_{mn}), \tag{2.64}$$

$$\{a_{m\alpha}, a_{n\beta}\} = 0, \quad \{b_{m\alpha}, b_{n\beta}\} = 0, \quad \{(a_{m\alpha})^\dagger, (a_{n\beta})^\dagger\} = 0, \quad \{(b_{m\alpha})^\dagger, (b_{n\beta})^\dagger\} = 0, \tag{2.65}$$

$$\{a_{m\alpha}, b_{n\beta}\} = 0, \quad \{(a_{m\alpha})^\dagger, (b_{n\beta})^\dagger\} = 0, \quad \{a_{m\alpha}, (b_{n\beta})^\dagger\} = 0, \quad \{(a_{m\alpha})^\dagger, b_{n\beta}\} = 0. \tag{2.66}$$

The left hand side of the first two equations is positive definite, so is the right, which leads to

$$2m \geqslant z^l, \quad \text{for } l = 1, \cdots, \mathrm{rank}\, Z. \tag{2.67}$$

This is the BPS bound condition eq. (2.41). Especially massless states can not have nonzero central charge, this is why we consider the massive representations here.

If central charges satisfy the condition

$$z^i = 2m \quad \text{for } i = 1, \cdots, r \leqslant \mathrm{rank}\, Z, \tag{2.68}$$

the corresponding operators b_i, $(b_i)^\dagger$ vanish. Then we have total of $2(N-r)$ sets of creation-annihilation operators. The irreducible representations constructed by such operators on a Clifford vacuum with spin j (or helicity λ) have total of $2^{2(N-r)}$ states and the maximum spin (or helicity) is $(N-r)/2 + j$ (or $(N-r) + \lambda$). If $r = \mathrm{rank}(Z) = N/2$ (for N even) the representations are the same as those in the massless case. Note in particular for a system with some central charges the maximum spin (helicity) can be consistently reduced.

3 Twisted Supersymmetry in Four Dimensions

As a variant of the notion of supersymmetry, one could consider "twisted" supersymmetry. Since, as noted above, this subject serves one of the main motivations to this article, let us here examine the technical aspect of twisted supersymmetry.

3.1 A Historical Review

Twisted supersymmetry was first proposed by Witten [4, 5] in the context of topological field theory (TFT). Topological field theories are independent on metrics of the base manifolds on which those theories are defined so that are only sensitive to topological structures of the manifolds. Witten showed that, firstly, so called the Donaldson invariants on a four manifold [17] can be computed within the framework of a quantum field theory, which we call the Donaldson-Witten theory, and, secondly, the Donaldson-Witten theory can be obtained by twisting $N = 2$ super Yang-Mills theory. Shortly after the Witten's work, that the twisted supersymmetry could be interpreted as a kind of BRST symmetry so that the Donaldson-Witten theory is derived by a BRST quantization of topological Yang-Mills theory in four dimensions [6, 7, 8, 9]. These facts implies that as twisted supersymmetric theories one might construct some topological models. Actually, for instance, topological matters in two [18, 19] and four [20] dimensions were shown to be constructed as twisted $N = 2$ supersymmetric theories in two and four dimensions, respectively. These observations were further generalized to twisting of $N = 4$ supersymmetric theories. Yamron introduced [21] two inequivalent types of twisting $N = 4$ super Yang-Mills

theories, and pointed out that one more type of twisting would be possible. Then Vafa and Witten constructed [22] a topological theory as a twisted Yang-Mills theory with twisting procedure equivalent to one of two introduced by Yamron. Yamron's third type of twist was later considered by Marcus [23] and Blau and Thompson [24]. These works were analyzed in detail from the viewpoint of the Mathai-Quillen formalism in [25].

We also remark that one alternative twisting of supersymmetry was introduced in [26, 27, 28, 29]. There twisted supersymmetry is strongly motivated by the Dirac-Kähler mechanism, and in fact the twisting process can be understood to be essentially the same as the Dirac-Kähler fermion formulation. We thus call this twist the Dirac-Kähler twist. Here we emphasize that the Dirac-Kähler twist, in four dimensions, is equivalent to the Yamron's third type as we will see later and, in two dimensions, to the two-dimensional twist mentioned above, although the motivations by which these twisting procedures are considered might be quite different.

In the following, we briefly review how these kinds of twisted supersymmetry in four dimensions can be defined, and list the corresponding twisted superalgebras.

3.2 General Twisting Procedure

Twist of supersymmetry can be understood to be the process of identifying (some part of) the internal symmetry (R-symmetry) with (some part of) the spacetime symmetry (Lorentz symmetry) and then defining a new representation of spacetime symmetry for the original supercharges. Such a process may include breaking of the internal symmetry group into some subgroups, one of which is isomorphic to (a subgroup of) the Lorentz group, and then taking the diagonal sum[9] of those of the isomorphic subgroups.

To be more specific, let us consider a supersymmetric theory with internal symmetry group $G_{\rm internal}$ and spacetime symmetry group $G_{\rm spacetime}$. For an N-extended supersymmetric theory in four dimensions, these symmetry groups should be taken as $SU(N)$ (or $U(N)$) and $P(1,3)$ (Poincaré group), respectively. We assume that the spacetime symmetry group can be decomposed irreducibly as $G_{\rm spacetime} \cong G \times G'_{\rm spacetime}$. A twisting procedure for this system can be taken if there exists a subgroup G' of $G_{\rm internal}$ such that $G_{\rm internal} \cong G' \times G'_{\rm internal}$ and also $G' \cong G$ (isomorphic). However, such decomposion of the internal symmetry group may not be possible in general unless we break the group into its appropriate subgroup. Here, to illustrate the twisting process, we first break the internal symmetry as $G_{\rm internal} \to \tilde{G}_{\rm internal}$, and then assume $\tilde{G}_{\rm internal}$ can be decomposed as $\tilde{G}_{\rm internal} \cong G' \times G'_{\rm internal}$. Since $G' \cong G$, we denote the former simply by G from now on. We therefore have a supersymmetric theory with associated symmetries $G'_{\rm internal} \times G \times G \times G'_{\rm spacetime}$. Now we are at the position of twisting of this system. The process is described schematically in Figure 1 and consists of steps below:

(i) Identification of two representations of G. To do this, we have to take two equivalent irreducible representations $(\rho_{\rm I}, V_{\rm I})$ and $(\rho_{\rm S}, V_{\rm S})$ of the two G's, respectively.

(ii) Taking the tensor product representation of the two irreducible representations of G. More precisely—the original symmetry $G \times G$ should be represented by $(\rho \times \rho, V \otimes V)$, where since $(\rho_{\rm I}, V_{\rm I}) \cong (\rho_{\rm S}, V_{\rm S})$ we simply denote them by (ρ, V). Then replace this by the tensor representation $(\rho \otimes \rho, V \otimes V)$. We also denote $G \times G$ simply by G' and understand that it is represented by the tensor product representation as above. Note here that G' is nothing but G; we just rename it just for the readability. Corresponding algebra now becomes $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{g}$, which should be represented by $(d\rho \otimes \mathbf{1} + \mathbf{1} \otimes d\rho, V \otimes V)$, or simply by $(d\rho + d\rho, V \otimes V)$—.

[9] More precisely, the diagonal sum of representations of the algebras corresponding to the internal symmetry and the spacetime symmetry. The corresponding representation space is the tensor product of the original representation spaces.

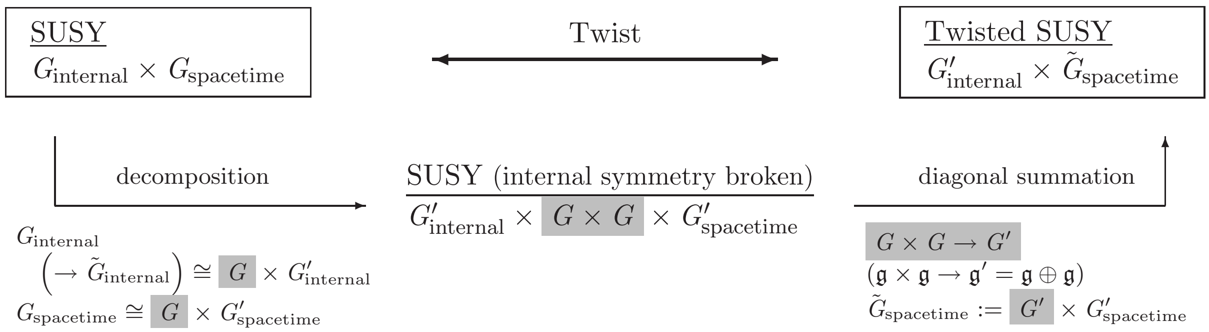

Figure 1: Schematic description of twisting process

(iii) Decompoion of the representation into irreducible representations. This step is equivalent to the Clebsch-Gordan decomposition of the tensor product representation $(\rho \otimes \rho, V \otimes V)$, or $(d\rho + d\rho, V \otimes V)$.

(iv) Redefinition of the spacetime symmetry by $\tilde{G}_{\text{spacetime}} := G' \times G'_{\text{spacetime}}$. Since $G' \equiv G$, this definition certainly restore the spacetime symmetry group though its representation is changed from the original one.

Through all these steps, we now obtain a system which has the twisted supersymmetry with associated symmetries $G'_{\text{internal}} \times \tilde{G}_{\text{spacetime}}$. One of the main outcome of such twisting process is that one can extract explicitly the scalar part of supercharges of the original supersymmetry as the singlet representation in the Clebsch-Gordan decomposition in the step (iii) above. In the context of TFT, such scalar supercharge can be interpreted as a topological charge which assures the topological nature of those theories, for, the scalar charge can be introduced on an arbitrary manifold (because it is a scalar), and the charge is nilpotent (if there is no central charge). We review some specific examples of twisting process of extended supersymmetries in four dimensions.

3.3 Twisted $N = 2$ Supersymmetry

First consider the twist of $N = 2$ supersymmetry in four-dimensional Euclidean spacetime, with R-symmetry group $U(2) \cong SU(2)_{\mathrm{I}} \times U(1)$ and Lorentz symmetry group $SO(4) \cong SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}}$. The labels I, L, R specify the representations. We denote the R-symmetry generators by R_{ij} and the Lorentz generators by $J_{\alpha\beta},\ J_{\dot{\alpha}\dot{\beta}}$, where

$$J_{\alpha\beta} := \frac{1}{4}(\sigma^{\mu\nu})_{\alpha\beta} J_{\mu\nu}, \quad J_{\dot{\alpha}\dot{\beta}} := \frac{1}{4}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} J_{\mu\nu} \tag{3.1}$$

correspond to the decomposition above. The superalgebra is given as (2.1)–(2.8), namely

$$\{Q_{i\alpha}, Q_{j\beta}\} = C_{ij} C_{\alpha\beta} Z, \qquad \{\overline{Q}^i{}_{\dot{\alpha}}, \overline{Q}^j{}_{\dot{\beta}}\} = C^{ij} C_{\dot{\alpha}\dot{\beta}} Z, \tag{3.2}$$

$$\{Q_{i\alpha}, \overline{Q}^j{}_{\dot{\beta}}\} = \delta_i{}^j P_{\alpha\dot{\beta}}, \qquad [P_{\alpha\dot{\beta}}, P_{\gamma\dot{\delta}}] = 0, \tag{3.3}$$

$$[Q_{i\alpha}, P_{\gamma\dot{\delta}}] = 0, \qquad [\overline{Q}^i{}_{\dot{\alpha}}, P_{\gamma\dot{\delta}}] = 0, \tag{3.4}$$

$$[J_{\alpha\beta}, Q_{i\gamma}] = \frac{i}{2} C_{\gamma(\alpha} Q_{i\beta)}, \qquad [J_{\alpha\beta}, \overline{Q}^i{}_{\dot{\gamma}}] = 0, \tag{3.5}$$

$$[J_{\dot{\alpha}\dot{\beta}}, Q_{i\gamma}] = 0, \qquad [J_{\dot{\alpha}\dot{\beta}}, \overline{Q}^i{}_{\dot{\gamma}}] = \frac{i}{2} C_{\gamma(\dot{\alpha}} \overline{Q}^i{}_{\dot{\beta})}, \tag{3.6}$$

$$[J_{\alpha\beta}, P_{\gamma\dot{\delta}}] = \frac{i}{2} C_{\gamma(\alpha} P_{\beta)\dot{\delta}}, \qquad [J_{\dot{\alpha}\dot{\beta}}, P_{\gamma\dot{\delta}}] = \frac{i}{2} C_{\dot{\delta}(\dot{\alpha}} P_{\gamma\dot{\beta})}, \tag{3.7}$$

$$[J_{\alpha\beta}, J_{\gamma\delta}] = \frac{i}{2} C_{(\alpha|(\gamma} J_{\delta)|\beta)}, \qquad [J_{\dot\alpha\dot\beta}, J_{\dot\gamma\dot\delta}] = \frac{i}{2} C_{(\dot\alpha|(\dot\gamma} J_{\dot\delta)|\dot\beta)}, \tag{3.8}$$

$$[J_{\alpha\beta}, J_{\dot\gamma\dot\delta}] = 0, \qquad [Z, \mathrm{any}] = 0, \tag{3.9}$$

$$[R_{ij}, Q_{k\gamma}] = \frac{i}{2} C_{k(i} Q_{j)\gamma}, \qquad [R_{ij}, \overline{Q}_{k\gamma}] = \frac{i}{2} C_{k(i} \overline{Q}_{j)\gamma}, \tag{3.10}$$

$$[R_{ij}, R_{kl}] = \frac{i}{2} C_{(i|(k} R_{l)|j)}, \qquad [R_{ij}, P_{\alpha\dot\beta}] = 0, \tag{3.11}$$

$$[R_{ij}, J_{\alpha\beta}] = 0, \qquad [R_{ij}, J_{\dot\alpha\dot\beta}] = 0, \tag{3.12}$$

where $P_{\alpha\dot\beta} := (\sigma^\mu)_{\alpha\dot\beta} P_\mu$. Supercharges $Q_{i\alpha}$ and $\overline{Q}^i{}_{\dot\alpha}$ have the $U(1)$ charges[10] $+1$ and -1, respectively, and the other generators have charges 0. Notice here that we have taken the representation of the R-symmetry to be equivalent to the representations of $SU(2)_{\mathrm{L,R}}$ of the Lorentz symmetry. Indices $i,\ j$ can be raised or lowered by the $SU(2)$ invariant tensors $C_{ij},\ C^{ij}$.

Then according to the procedure above, we first identify $SU(2)_{\mathrm{I}}$ as either of $SU(2)_{\mathrm{L}}$ or $SU(2)_{\mathrm{R}}$, and after that, we take the tensor product representation. In this case, these process should be described as the identification of indices i as either of α or $\dot\alpha$, and then as taking the diagonal summation

$$J'_{\alpha\beta} := J_{\alpha\beta} + R_{\alpha\beta}, \quad \text{or} \quad J'_{\dot\alpha\dot\beta} := J_{\dot\alpha\dot\beta} + R_{\dot\alpha\dot\beta}. \tag{3.13}$$

We represent these twisting prescription simply as

$$\begin{array}{ccc} \mathbf{2} & \to & (\mathbf{2},\mathbf{1}) \\ i & \to & \alpha \end{array}, \quad \text{or} \quad \begin{array}{ccc} \mathbf{2} & \to & (\mathbf{1},\mathbf{2}) \\ i & \to & \dot\alpha \end{array}. \tag{3.14}$$

In what follows, we only treats the former diagonal summation since these two are obviously equivalent, so that the tensor product representation can be expressed as $Q_{\alpha\beta},\ \overline{Q}_{\alpha\dot\beta}$. Next, we take the Clebsch-Gordan decomposition

$$\begin{array}{ccccc} \mathbf{2}\otimes\mathbf{2} & = & \mathbf{1} & \oplus & \mathbf{3} \\ Q_{\alpha\beta} & \to & Q_{[\alpha\beta]}, & & Q_{(\alpha\beta)}, \end{array} \tag{3.15}$$

and, for simplicity, we denote as

$$Q := \frac{1}{2} C^{\alpha\beta} Q_{[\alpha\beta]}, \quad H_{\alpha\beta} := Q_{(\alpha\beta)}, \quad G_{\alpha\dot\beta} := \overline{Q}_{\alpha\dot\beta}. \tag{3.16}$$

The remaining task is to define a new representation of Lorentz symmetry. This can be realized, as will be clear shortly, just by considering the symmetry to be generated by $J'_{\alpha\beta}$ and $J_{\dot\alpha\dot\beta}$.

Thus we complete a twisting of the $N=2$ supersymmetry. The resultant twisted $N=2$ superalgebra is readily derived from the original superalgebra and the definitions of the twisted supercharges above. Renaming $J'_{\alpha\beta}$ simply as $J_{\alpha\beta}$, we obtain

$$\{Q, Q\} = Z, \qquad \{H_{\alpha\beta}, H_{\gamma\delta}\} = C_{(\alpha|(\gamma} C_{\delta)|\beta)} Z, \tag{3.17}$$

$$\{G_{\alpha\dot\beta}, G_{\gamma\dot\delta}\} = C_{\alpha\beta} C_{\dot\beta\dot\delta} Z, \qquad \{Q, H_{\alpha\beta}\} = 0, \tag{3.18}$$

$$\{Q, G_{\alpha\dot\beta}\} = P_{\alpha\dot\beta}, \qquad \{H_{\alpha\beta}, G_{\gamma\dot\delta}\} = C_{\alpha(\gamma} P_{\beta)\dot\delta}, \tag{3.19}$$

$$[Q, P_{\gamma\dot\delta}] = 0, \qquad [H_{\alpha\beta}, P_{\gamma\dot\delta}] = 0, \tag{3.20}$$

$$[G_{\alpha\dot\beta}, P_{\gamma\dot\delta}] = 0, \qquad [P_{\alpha\dot\beta}, P_{\gamma\dot\delta}] = 0, \tag{3.21}$$

$$[J_{\alpha\beta}, Q] = 0, \qquad [J_{\dot\alpha\dot\beta}, Q] = 0, \tag{3.22}$$

$$[J_{\alpha\beta}, H_{\gamma\delta}] = \frac{i}{2} C_{(\alpha|(\gamma} H_{\delta)|\beta)}, \qquad [J_{\dot\alpha\dot\beta}, H_{\gamma\delta}] = 0, \tag{3.23}$$

[10] More precisely, we only have $U(1)/Z_4$ symmetry unless the central charge vanishes.

$$[J_{\alpha\beta}, G_{\gamma\dot{\delta}}] = \frac{i}{2} C_{(\alpha|\gamma} G_{\beta)\dot{\delta}}, \qquad [J_{\dot{\alpha}\dot{\beta}}, G_{\gamma\dot{\delta}}] = \frac{i}{2} C_{(\dot{\alpha}|\dot{\delta}} G_{\gamma\dot{\beta})}, \tag{3.24}$$

$$[J_{\alpha\beta}, P_{\gamma\dot{\delta}}] = \frac{i}{2} C_{(\alpha|\gamma} P_{\beta)\dot{\delta}}, \qquad [J_{\dot{\alpha}\dot{\beta}}, P_{\gamma\dot{\delta}}] = \frac{i}{2} C_{(\dot{\alpha}|\dot{\delta}} P_{\gamma\dot{\beta})}, \tag{3.25}$$

$$[J_{\alpha\beta}, J_{\gamma\delta}] = \frac{i}{2} C_{(\alpha|(\gamma} J_{\delta)|\beta)}, \qquad [J_{\dot{\alpha}\dot{\beta}}, J_{\dot{\gamma}\dot{\delta}}] = \frac{i}{2} C_{(\dot{\alpha}|(\dot{\gamma}} J_{\dot{\delta})|\dot{\beta})}, \tag{3.26}$$

$$[J_{\alpha\beta}, J_{\dot{\gamma}\dot{\delta}}] = 0, \qquad [Z, \text{any}] = 0. \tag{3.27}$$

Note here that $J_{\alpha\beta}$ and $J_{\dot{\alpha}\dot{\beta}}$ can actually interpreted as the generators of a new Lorentz group. The twisted charges Q, $H_{\alpha\beta}$, $G_{\alpha\dot{\beta}}$ transform under this new Lorentz symmetry as if they are a scalar, a (anti-)selfdual tensor, and a vector, as implied by their indices, and they have the $U(1)$ charges $+1$, $+1$, -1, respectively.

We could have taken, as mentioned above, another diagonal summation based on the identification $i \to \dot{\alpha}$. The result is almost the same as obtained above, except that the role of selfduality and anti-selfduality should be exchanged. Thus we find that there is essentially one twist for $N = 2$ supersymmetry in four dimensions.

Since we have obtain the twisted superalgebra, a twisted supersymmetric field theory can be constructed based on the algebra. However, we can find such a theory by directly twisting a theory with the ordinary $N = 2$ supersymmetry. As an example, consider the $N = 2$ super Yang-Mills theory. It contains a gauge vector $A_{\alpha\dot{\beta}}$, Weyl spinors $\psi_{i\alpha}$, $\overline{\psi}^{i}{}_{\dot{\alpha}}$, complex scalars ϕ, $\overline{\phi}$, and auxiliary fields G_{ij}. The twist above can be applied to this theory and leads to the twisted field contents

$$A_{\alpha\dot{\beta}} \to A_{\alpha\dot{\beta}}, \tag{3.28}$$

$$\lambda_{i\beta} \to \lambda_{\alpha\beta} \to \eta,\ \chi_{\alpha\beta}, \tag{3.29}$$

$$\overline{\lambda}_{i\dot{\beta}} \to \psi_{\alpha\dot{\beta}}, \tag{3.30}$$

$$\phi \to \phi, \tag{3.31}$$

$$\overline{\phi} \to \rho, \tag{3.32}$$

$$G_{ij} \to G_{\alpha\beta}, \tag{3.33}$$

where $\eta = C^{\alpha\beta}\psi_{\alpha\beta}$, $\chi_{\alpha\beta} = \psi_{(\alpha\beta)}$, etc. The action and supertransformations can also be twisted based on these (re-)definitions of fields and supercharges.

3.4 Twisted $N = 4$ Supersymmetries

We now review the survey of possible twists of $N = 4$ supersymmetry in Euclidean four-dimensional spacetime.

Breaking of the Internal Symmetries According to the general procedure for twisting explained above, we have to first take a representation of a subgroup of the R-symmetry which can be identified as a representation of (a subgroup of) the Lorentz symmetry. In Euclidean four-dimensional spacetime, the Lorentz symmetry should be given by $SO(4) \cong SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}}$ as before. For an $N = 4$ supersymmetric theory, the full R-symmetry should be $SU(4)$. Since $SU(4)$ has no subgroup which can factorize directly as in $SU(4) \cong G \times G'$, we have to break the $SU(4)$ into some suitable subgroup which is then factorized by $SU(2)$ or $SU(2) \times SU(2) \cong SO(4)$ corresponding to the factorization of the Lorentz symmetry.

Actually, such nontrivial decomposition of $SU(4)$ is determined uniquely once we choose an $SU(2)$ subgroup to be factorized, since remaining factors must be the (maximal) subgroup which makes this chosen $SU(2)$ factor to be an invariant subgroup. We have essentially two

possibilities:

$$\begin{cases} SU(4) \to SU(2) \times SU(2), \\ \mathfrak{su}(4) \to \mathfrak{su}(2) \oplus \mathfrak{su}(2) = \left\{ \sqrt{1/2} \left(X^{12}_{\mathrm{s,a}} + X^{34}_{\mathrm{s,a}} \right),\ \sqrt{1/2} X^1 + \sqrt{1/6} \left(\sqrt{2} X^3 - X^2 \right) \right\} \\ \qquad\qquad\qquad\qquad\qquad \oplus \left\{ \sqrt{1/2} \left(X^{13}_{\mathrm{s,a}} + X^{24}_{\mathrm{s,a}} \right),\ \sqrt{1/3} \left(\sqrt{2} X^2 + X^3 \right) \right\}, \end{cases} \tag{3.34}$$

$$\begin{cases} SU(4) \to SU(2) \times SU(2) \times U(1), \\ \mathfrak{su}(4) \to \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \tilde{\mathfrak{u}}(1) = \left\{ X^{12}_{\mathrm{s,a}},\ X^1 \right\} \oplus \left\{ X^{34}_{\mathrm{s,a}},\ \sqrt{1/3} \left(\sqrt{2} X^3 - X^2 \right) \right\} \\ \qquad\qquad\qquad\qquad\qquad \oplus \left\{ \sqrt{1/3} \left(\sqrt{2} X^2 + X^3 \right) \right\}, \end{cases} \tag{3.35}$$

where the $SU(N)$ generators X, with $N = 4$, are defined by (A.67) and (A.68). These decompositions are further restricted by specifying representations of each $SU(2)$ factor. In order to carry out twisting, the former should be assigned as

$$SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{I}}, \quad \text{or equivalently,} \quad SU(2)_{\mathrm{R}} \times SU(2)_{\mathrm{I}}, \tag{3.36}$$

which corresponds to

$$\begin{cases} \mathbf{4} \to (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}), \\ i \to a\alpha, \end{cases} \quad \text{or equivalently,} \quad \begin{cases} \mathbf{4} \to (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}), \\ i \to a\dot{\alpha}, \end{cases} \tag{3.37}$$

while the latter should be assigned as

$$SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{I}} \times U, \quad \text{or equivalently,} \quad SU(2)_{\mathrm{R}} \times SU(2)_{\mathrm{I}} \times U, \tag{3.38}$$

or

$$SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}} \times U, \tag{3.39}$$

which corresponds to

$$\begin{cases} \mathbf{4} \to (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}), \\ i \to (\alpha, a) \equiv a \oplus \alpha, \end{cases} \quad \text{or equivalently,} \quad \begin{cases} \mathbf{4} \to (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}), \\ i \to (\dot{\alpha}, a) \equiv \dot{\alpha} \oplus a, \end{cases} \tag{3.40}$$

or

$$\begin{cases} \mathbf{4} \to (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}), \\ i \to (\alpha, \dot{\alpha}) \equiv \alpha \oplus \dot{\alpha}. \end{cases} \tag{3.41}$$

These three (out of five) decompositions of representations provide inequivalent possible twisting of $N = 4$ supersymmetry [21, 25]. Let us now examine how supercharges are represented after each of these twisting processes.

Vafa-Witten Twist This is the twist for the decomposition [21, 22]

$$SU(4) \to SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{I}}, \quad \begin{cases} \mathbf{4} \to (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}), \\ i \to a\alpha. \end{cases} \tag{3.42}$$

Diagonal summation is taken w.r.t. this $SU(2)_{\mathrm{L}}$ and the one in the Lorentz group. The resultant twisted theory thus has the symmetry group $SU(2)_{\mathrm{I}} \times SU(2)_{\mathrm{L}'} \times SU(2)_{\mathrm{R}}$. Supercharges are twisted as

$$\begin{cases} Q_{i\beta} \to Q_{a\,\alpha,\beta} \to Q_a,\ Q_{a\alpha\beta}, \\ \overline{Q}^i{}_{\dot{\beta}} \to Q^a_{\alpha\dot{\beta}}, \end{cases} \tag{3.43}$$

while the on-shell $N=4$ super Yang-Mills multiplet is twisted as

$$A_{\alpha\dot{\beta}} \to A_{\alpha\dot{\beta}}, \tag{3.44}$$

$$\lambda_{i\beta} \to \lambda_{a\,\alpha,\beta} \to \eta_a,\ \chi_{a\,\alpha\beta}, \tag{3.45}$$

$$\overline{\lambda}^i{}_{\dot{\beta}} \to \psi^a{}_{\alpha\dot{\beta}}, \tag{3.46}$$

$$\phi_{ij} \to \phi_{a\alpha,b\beta} \to \varphi_{ab},\ G_{\alpha\beta}, \tag{3.47}$$

where ${}_{ab}$ and ${}_{\alpha\beta}$ denote symmetric labeling w.r.t. $a \leftrightarrow b$ and $\alpha \leftrightarrow \beta$, respectively. Note here that $\phi_{ij}+\phi_{ji}=0$ so that $\phi_{a\alpha,b\beta}$ must be antisymmetric w.r.t. $a \leftrightarrow b$ ($\alpha \leftrightarrow \beta$) and at the same time be symmetric w.r.t. $\alpha \leftrightarrow \beta$ ($a \leftrightarrow b$, respectively).

Half Twist This corresponds to [21]

$$SU(4) \to SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{I}} \times U(1), \quad \begin{cases} \mathbf{4} \to (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}), \\ i \to (a,\alpha) \equiv a \oplus \alpha. \end{cases} \tag{3.48}$$

Diagonal summation is taken for this $SU(2)_{\mathrm{L}}$, so the resultant symmetry group is $SU(2)_{\mathrm{I}} \times U(1) \times SU(2)_{\mathrm{L}'} \times SU(2)_{\mathrm{R}}$. Twisted supercharges are then

$$\begin{cases} Q_{i\beta} \to Q_{a\oplus\alpha,\beta} \to Q^{(-1)}_{a\beta},\ Q^{(+1)},\ Q^{(+1)}_{\alpha\beta}, \\ \overline{Q}^i{}_{\dot{\beta}} \to Q^{a\oplus\alpha}{}_{\dot{\beta}} \to Q^{(+1)}_{a\dot{\beta}},\ Q^{(-1)}_{\alpha\dot{\beta}}, \end{cases} \tag{3.49}$$

where we have denoted the $U(1)$ charges[11] in the superscripts. The $N=4$ on-shell SYM multiplet is twisted as

$$A_{\alpha\dot{\beta}} \to A^{(0)}_{\alpha\beta}, \tag{3.50}$$

$$\lambda_{i\beta} \to \lambda_{a\oplus\alpha\,\beta} \to \lambda^{(+1)}_{a\beta},\ \eta^{(-1)},\ \chi^{(-1)}_{\alpha\beta}, \tag{3.51}$$

$$\overline{\lambda}^i{}_{\dot{\beta}} \to \overline{\lambda}^{a\oplus\alpha}{}_{\dot{\beta}} \to \psi^{(+1)}_{\alpha\dot{\beta}},\ \zeta^{(-1)\,a}_{\dot{\beta}}, \tag{3.52}$$

$$\phi_{ij} \to \phi_{a\oplus\alpha,b\oplus\beta} \to B^{(-2)},\ C^{(+2)},\ G^{(0)}_{a\alpha} (= \phi_{a\beta},\ \phi_{b\alpha}). \tag{3.53}$$

Amphicheiral Twist The last one is for [21, 23, 24]

$$SU(4) \to SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}} \times U(1), \quad \begin{cases} \mathbf{4} \to (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}), \\ i \to (\alpha,\dot{\beta}) \equiv \alpha \oplus \dot{\beta}. \end{cases} \tag{3.54}$$

The twisted supercharges are

$$\begin{cases} Q_{i\gamma} \to Q_{\alpha\oplus\dot{\beta},\gamma} \to Q^{(+1)},\ Q^{(-1)}_{\gamma\dot{\beta}},\ Q^{(+1)}_{\alpha\gamma}, \\ \overline{Q}^i{}_{\dot{\gamma}} \to \overline{Q}^{\alpha\oplus\dot{\beta}}{}_{\dot{\gamma}} \to \tilde{Q}^{(+1)},\ \tilde{Q}^{(-1)}_{\alpha\dot{\gamma}},\ \tilde{Q}^{(+1)}_{\dot{\beta}\dot{\gamma}}. \end{cases} \tag{3.55}$$

Thus we find that such theory contains two scalar supercharges with the same $U(1)$ charges. The $N=4$ SYM multiplet is now twisted as

$$A_{\alpha\dot{\beta}} \to A^{(0)}_{\alpha\dot{\beta}}, \tag{3.56}$$

$$\lambda_{i\gamma} \to \lambda_{\alpha\oplus\dot{\beta}\,\gamma} \to \tilde{\psi}^{(+1)}_{\gamma\dot{\beta}},\ \eta^{(-1)},\ \chi^{(-1)}_{\alpha\gamma}, \tag{3.57}$$

$$\overline{\lambda}^i{}_{\dot{\gamma}} \to \overline{\lambda}^{\alpha\oplus\dot{\beta}}{}_{\dot{\gamma}} \to \psi^{(+1)}_{\alpha\dot{\gamma}},\ \tilde{\eta}^{(-1)},\ \tilde{\chi}^{(-1)}_{\dot{\beta}\dot{\gamma}}, \tag{3.58}$$

$$\phi_{ij} \to \phi_{\alpha\oplus\dot{\beta},\gamma\oplus\dot{\delta}} \to B^{(-2)},\ C^{(+2)},\ V^{(0)}_{\alpha\dot{\beta}}. \tag{3.59}$$

[11] These charges can be assigned essentially as the eigenvalues of $\sqrt{2}X^2 + X^3$.

3.5 Dirac-Kähler Twist

Let us now consider one more type of twist which we call the Dirac-Kähler twist (or the DK twist for short) [26, 27, 28, 29]. As noted above, there are essentially three different types of twist of $N = 4$ supersymmetry in four dimensions. Thus, the DK twist is expected to be equivalent to one of the three. In fact, the twist, in four dimensions, will be recognized as the amphicheiral twist listed at the end of the preceding section.

In principle, the DK twist can be defined in any dimensions. Here again we consider Euclidean spacetime. In Euclidean spacetime with dimension D, spacetime symmetry group should be $SO(D) \cong Spin(D)$, where $Spin(D)$ is the double (and indeed the universal) cover of $SO(D)$. As for R-symmetry group G_{I}, the DK twist restricts it to be $Spin(D)$, or, more generally, to be a group including $Spin(D)$ as a subgroup. The DK twist is the process of taking the diagonal sum of these two $Spin(D)$, one is the Lorentz group and the other is the R-symmetry group, and then identifying the resultant $Spin(D)$ as a new Lorentz symmetry group.

To be specific, consider supercharges $Q_{A\alpha}$ and $\overline{Q}^{A\alpha}$ in D-dimensional Euclidean spacetime with R-symmetry $G_{\mathrm{I}} = Spin(D)$. For simplicity, we work on the case where D is even. Then A and α are labels of $2^{D/2}$-component spinors. The Dirac conjugate is defined as[12]

$$\overline{Q}^{A\alpha} = \sum_{B\beta} (Q_{B\beta})^\dagger (\Gamma^0)_B{}^A (\Gamma^0)_\beta{}^\alpha. \tag{3.60}$$

We also assume that the $Spin(D)$-Majorana (or $Spin(D)$-Majorana-Weyl, if possible) conditions are imposed on the supercharges, as in

$$Q_{A\alpha} = (Q_{\mathrm{c}})_{A\alpha} := C_{AB} C_{\alpha\beta} (\overline{Q}^T)^{B\beta}, \tag{3.61}$$

where the charge conjugation matrices are such that

$$C^T = \varepsilon' C, \quad C^\dagger C = 1, \quad C\gamma^\mu C^{-1} = \eta' (\gamma^\mu)^T, \quad C\gamma^i C^{-1} = \eta'(\gamma^i)^T, \quad \varepsilon',\ \eta' = \pm 1. \tag{3.62}$$

The DK twist now reads

$$G_{\mathrm{I}} \to Spin(D), \quad \begin{cases} \mathbf{2^{D/2}} \to \mathbf{2^{D/2}}, \\ A \to \alpha, \\ Q_{A\beta} \to Q_{\alpha\beta}. \end{cases} \tag{3.63}$$

Using eq. (B.9), we obtain a Clebsch-Gordan decomposition as

$$Q_{\alpha\beta} = \sum_{p=0}^{D} \frac{1}{p!} Q_{\mu_1 \cdots \mu_p} (\gamma^{\mu_1 \cdots \mu_p} C^{-1})_{\alpha\beta}, \quad Q_{\mu_1\cdots\mu_p} := (-1)^{p(p-1)/2} \frac{1}{2^{D/2}} (C\gamma_{\mu_1\cdots\mu_p})^{\alpha\beta} Q_{\alpha\beta}. \tag{3.64}$$

We call $Q_{\mu_1\cdots\mu_p}$ the DK twisted supercharges. Note that they contain particularly the scalar charge $Q := \frac{1}{2^{D/2}} C^{\alpha\beta} Q_{\alpha\beta}$ for $p = 0$ as well as the pseudo scalar charge $\tilde{Q} := \frac{1}{2^{D/2}} (C\Gamma^5)^{\alpha\beta} Q_{\alpha\beta}$ for $p = D$. Note also that by taking the linear combinations

$$\begin{aligned} Q^\pm_{\mu_1\cdots\mu_p} &:= \frac{1}{2}\left(Q_{\mu_1\cdots\mu_p} \pm \frac{1}{(D-p)!} (-1)^{D/4 + p(p+1)/2} \varepsilon_{\mu_1\cdots\mu_p}{}^{\mu_{p+1}\cdots\mu_D} Q_{\mu_{p+1}\cdots\mu_D} \right) \\ &= \frac{1}{2^{D/2}} (-1)^{p(p-1)/2} \left(C\gamma_{\mu_1\cdots\mu_p} \frac{1}{2}\left(1 \pm \Gamma^5\right) \right)^{\alpha\beta} Q_{\alpha\beta}, \end{aligned} \tag{3.65}$$

we can explicitly denote the chiralities of the supercharges.

[12]See appendix B for these conventions.

The corresponding superalgebra should be

$$\{Q_{A\alpha}, Q_{B\beta}\} = 2C^{-1}_{AB}(\gamma^\mu C^{-1})_{\alpha\beta}P_\mu + \Gamma_{\alpha\beta}Z_{AB}, \tag{3.66}$$

$$[J_{\mu\nu}, Q_{A\alpha}] = \frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_{A\beta}, \qquad [R_{ij}, Q_{A\alpha}] = \frac{i}{2}(\gamma_{ij})_A{}^B Q_{B\alpha}, \tag{3.67}$$

where, just for simplicity, we have chosen $\eta' = 1$ so that C^{-1}_{AB} and $(\gamma_\mu C^{-1})_{\alpha\beta}$ should become both antisymmetric or both symmetric, and $\Gamma_{\alpha\beta}$ is restricted to be $C^{-1}_{\alpha\beta}$ or $(\Gamma^5 C^{-1})_{\alpha\beta}$ or both by the Coleman-Mandula theorem. The DK twisted algebra can be readily obtained using eq. (B.6) and the definition of the twisted supercharges. For example, in a simple case $\Gamma_{\alpha\beta}Z_{AB} = C_{\alpha\beta}C_{AB}Z$, we have

$$\{Q_{\mu_1\cdots\mu_p}, Q_{\nu_1\cdots\nu_{p-1}}\} = \frac{2}{2^{D/2}}\eta_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_p}P^{\nu_p}, \quad \{Q_{\mu_1\cdots\mu_p}, Q_{\nu_1\cdots\nu_p}\} = \frac{1}{2^{D/2}}\eta_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_p}Z, \tag{3.68}$$

$$\begin{aligned}[J_{\mu\nu}, Q_{\rho_1\cdots\rho_p}] = \frac{i}{2}\varepsilon'(-1)^{p(p-1)/2}\Big(&Q_{\mu\nu\rho_1\cdots\rho_p} + \sum_{i=1}^{p}\eta_{[\mu|\rho_i}Q_{\rho_1\cdots\nu]\cdots\rho_p} \\ &+ \sum_{1\leqslant i<j\leqslant p}(-1)^{i+j}(\eta_{[\mu|\rho_i}\eta_{\nu]\rho_j})Q_{\rho_1\cdots\breve{\rho}_i\cdots\breve{\rho}_j\cdots\rho_p}\Big),\end{aligned} \tag{3.69}$$

$$\begin{aligned}[R_{\mu\nu}, Q_{\rho_1\cdots\rho_p}] = \frac{i}{2}\varepsilon'(-1)^{p(p-1)/2}\Big(&-Q_{\mu\nu\rho_1\cdots\rho_p} + \sum_{i=1}^{p}\eta_{[\mu|\rho_i}Q_{\rho_1\cdots\nu]\cdots\rho_p} \\ &- \sum_{1\leqslant i<j\leqslant p}(-1)^{i+j}(\eta_{[\mu|\rho_i}\eta_{\nu]\rho_j})Q_{\rho_1\cdots\breve{\rho}_i\cdots\breve{\rho}_j\cdots\rho_p}\Big).\end{aligned} \tag{3.70}$$

It is then crucial that by taking the diagonal summation $J'_{\mu\nu} := J_{\mu\nu} + R_{\mu\nu}$, we have

$$[J'_{\mu\nu}, Q_{\rho_1\cdots\rho_p}] = i\varepsilon'(-1)^{p(p-1)/2}\sum_{i=1}^{p}\eta_{[\mu|\rho_i}Q_{\rho_1\cdots\nu]\cdots\rho_p}, \tag{3.71}$$

which can be identified as the ordinary Lorentz transformations of a tensor quantity so that, in particular, $[J'_{\mu\nu}, Q] = 0$.

Finally let us take a closer look at the DK twist of $N = 4$ supersymmetry in Euclidean four-dimensional spacetime. The R-symmetry has to be $Spin(4) \cong SO(4)$. Twisted supercharges are defined as in (3.64). Here we redefine them, changing some trivial constant factors including sings, as

$$Q := C^{\alpha\beta}Q_{\alpha\beta}, \qquad Q_\mu := (C\gamma_\mu)^{\alpha\beta}Q_{\alpha\beta}, \qquad Q_{\mu\nu} := (C\gamma_{\mu\nu})^{\alpha\beta}Q_{\alpha\beta}, \tag{3.72}$$

$$\tilde{Q} := (C\gamma^5)^{\alpha\beta}Q_{\alpha\beta}, \quad \tilde{Q}_\mu := (C\gamma^5\gamma_\mu)^{\alpha\beta}Q_{\alpha\beta}, \quad \tilde{Q}_{\mu\nu} := (C\gamma^5\gamma_{\mu\nu})^{\alpha\beta}Q_{\alpha\beta}. \tag{3.73}$$

Now take specific representations

$$\gamma^\mu := \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = (\mathbf{1}, i\tau^i), \quad \bar{\sigma}^\mu = (\mathbf{1}, -i\tau^i), \tag{3.74}$$

so that

$$C := i\gamma^1\gamma^3 = \begin{pmatrix} -\tau^2 & 0 \\ 0 & -\tau^2 \end{pmatrix}, \quad \gamma^5 := \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.75}$$

Then in terms of the Weyl basis

$$\psi_\alpha \to \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \quad Q_{\alpha\beta} \to \begin{pmatrix} Q_{\alpha\beta} & Q_\alpha{}^{\dot{\beta}} \\ Q^{\dot{\alpha}}{}_\beta & Q^{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \tag{3.76}$$

we find that

$$Q = C^{\alpha\beta}_{(2)} Q_{\alpha\beta} + C^{(2)}_{\dot{\alpha}\dot{\beta}} Q^{\dot{\alpha}\dot{\beta}}, \qquad \tilde{Q} = C^{\alpha\beta}_{(2)} Q_{\alpha\beta} - C^{(2)}_{\dot{\alpha}\dot{\beta}} Q^{\dot{\alpha}\dot{\beta}}, \tag{3.77}$$

$$Q_\mu = -(\sigma_\mu)_{\alpha\dot{\beta}} Q^{\alpha\dot{\beta}} + (\bar{\sigma}_\mu)^{\dot{\alpha}\beta} Q_{\dot{\alpha}\beta}, \qquad \tilde{Q}_\mu = (\sigma_\mu)_{\alpha\dot{\beta}} Q^{\alpha\dot{\beta}} + (\bar{\sigma}_\mu)^{\dot{\alpha}\beta} Q_{\dot{\alpha}\beta}, \tag{3.78}$$

$$Q_{\mu\nu} = (\sigma_{\mu\nu})^{\alpha\beta} Q_{\alpha\beta} + (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} Q^{\dot{\alpha}\dot{\beta}}, \qquad \tilde{Q}_{\mu\nu} = (\sigma_{\mu\nu})^{\alpha\beta} Q_{\alpha\beta} - (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} Q^{\dot{\alpha}\dot{\beta}}. \tag{3.79}$$

Thus it is clear that they are essentially the same as supercharges in the amphicheiral twist above, for

$$Q_{[\alpha\beta]} = \frac{1}{2}(Q + \tilde{Q}) C^{(2)}_{\alpha\beta}, \qquad Q^{[\dot{\alpha}\dot{\beta}]} = \frac{1}{2}(Q - \tilde{Q}) C^{\dot{\alpha}\dot{\beta}}_{(2)}, \tag{3.80}$$

$$Q_{\alpha\dot{\beta}} = \frac{1}{4}(Q_\mu + \tilde{Q}_\mu)(\sigma^\mu)_{\alpha\dot{\beta}}, \qquad Q^{\dot{\alpha}\beta} = -\frac{1}{4}(Q_\mu - \tilde{Q}_\mu)(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}, \tag{3.81}$$

$$Q_{(\alpha\beta)} = \frac{1}{4}(Q_{\mu\nu} + \tilde{Q}_{\mu\nu})(\sigma^{\mu\nu})_{\alpha\beta}, \qquad Q^{(\dot{\alpha}\dot{\beta})} = \frac{1}{4}(Q_{\mu\nu} - \tilde{Q}_{\mu\nu})(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}}, \tag{3.82}$$

and the $U(1)$ charges in the amphicheiral twist can be identified as products of the chiralities w.r.t. to the two spinor indices, one originates from those for the R-symmetry and the other from those for the Lorentz symmetry. We therefore understand that the Dirac-Kähler twist for $N = 4$ supersymmetry in four dimensions is equivalent to the amphicheiral twist, though the former is applicable in any dimensions.

4 Superspace Formulation of $N = D = 4$ SYM with a Central Charge

In this section, a superspace formulation using superconnections and supercurvatures is applied to $N = 4$ super Yang-Mills theory with a central charge in four dimensions. There is no known superspace formulation for $N = 4$ supersymmetric theories with manifest supersymmetry. Thus in particular the $N = 4$ model, whose off-shell formulation is known [11, 12], is carefully surveyed to see how difficulties occur in the extension to its manifest supersymmetric superspace formulation.

4.1 A Historical Review

Super Yang-Mills theory (SYM) is defined to be a supersymmetric gauge theory which contains a vector gauge boson with spin 1 and has no field contents with higher spins. From the theoretical point of view, an off-shell formulation of such theories is quite important. Among others, superspace formulations, especially that using superconnections and supercurvatures on the superspace, should serve the most natural and powerful scheme to construct the off-shell super Yang-Mills theories.

Formulations of super Yang-Mills theories in four dimensions were developed long ago with $N = 1$ [30, 31, 32], for $N = 2$ [31, 32, 33] and for (on-shell) $N = 4$ [34] supersymmetries. Counterparts in two dimensions were also found as well [35, 36]. Later, systematic construction of supersymmetric gauge theories on superspace with superconnections and supercurvatures was proposed for both Abelian and non-Abelian gauge groups with $N = 1$ supersymmetries in four dimensions [37, 38]. Then it was generalized to superspace formulation for four-dimensional $N = 2$ super Yang-Mills theory [39] and also applied to many works, including, for example, the $N = 2$ twisted superspace formulation with twisted superconnections and supercurvatures of some topological field theories in four and two dimensions [20, 28]. However further generalization of such formulations simply to $N = 4$ super Yang-Mills theory based on the internal $SU(4)$

symmetry[13] has failed unless it breaks to the $N=2$ super Yang-Mills or merely results in an on-shell $N=4$ formulation [10]. In fact, no superspace formulation for the $SU(4)$ SYM which contains fields with spins only less than 1 has been constructed. Even an off-shell formulation for the $SU(4)$ model which includes the usual on-shell field contents, namely, 1 vector, 4 Majorana spinors and 6 real scalars [40], as well as complete lists of an auxiliary fields, has not been constructed. However, an $N=4$ supersymmetric gauge theory which contains higher spin fields as well as higher derivatives in an action has been formulated on twisted superspace in four dimensions [28]. This is the only known example of $N=4$ superspace formulation with manifest supersymmetry.

One of the essential difficulties to construct $N=4$ super Yang-Mills theory on superspace comes from the structure of the field contents of the multiplets in $N=4$ supersymmetry. As we have seen in section 2.3.2, massless representations, including vector multiplet for super Yang-Mills theory, of N-extended superalgebra has maximum helicity (spin) $\geqslant N/2$. Thus, an $N=4$ supersymmetric theory may contain the maximum helicity $\geqslant 4/2=2$. Especially the simple generalization of [39] to the $SU(4)$ SYM on superspace is based on the off-shell Clifford vacuum with helicity 0, so that it must contain fields with helicity >1 (up to 2) which are not allowed in the SYM multiplet. Then one may consider if one can consistently prohibit for such higher spin fields to emerge by imposing some suitable constraints. However no such procedure has succeeded [10] as noted above. This is one of the main reason one cannot complete superspace formulation of SYM with the usual field contents.

In [11, 12], it was shown that $N=4$ super Yang-Mills with another kind of $N=4$ SYM multiplet which includes a central charge in the superalgebra based on the internal $USp(4)$ symmetry can be formulated off-shell both for Abelian and non-Abelian gauge groups[14]. The basic idea is to introduce a central charge into the $N=4$ super Yang-Mills theory in order to reduce the possible maximum spin of the filed contents so that the theory contains spins less than 1. We have seen the mechanism how central charges reduce the on-shell spins (helicities). Such a mechanism to reduce the spins successfully works in their formulations. However, they did not construct the corresponding superspace formulation in completely closed in four dimensions.

There also proposed in [11, 12] off-shell, though not on superspace, $N=2$ super Yang-Mills theory with a central charge. Corresponding and applied superspace formulations were later developed with a $N=1$ vector superfield [41] and, by gauging the central charge [42, 43], with $N=2$ superconnections and supercurvatures [45] and with the harmonic superspace formalism [44], to be contrasted with the $N=2$ super Yang-Mills theory with ungaged central charge [46].

On the other hand, the $N=4$ super Yang-Mills theory ($USp(4)$ model) has not been constructed explicitly on superspace, though briefly mentioned in [12]. In the following, we will thus attempt to formulate the $USp(4)$ SYM with a central charge. Since it is formulated off-shell, it seems possible to develop the corresponding superspace formulation.

4.2 The $USp(4)$ Superalgebra

Let us now go on to the analysis of the $USp(4)$ model. First recall that, as noted in section 2, in order to introduce a central charge, we have to break the full internal symmetry $SU(4)$ into some automorphic subgroup. As such subgroup, we can adopt $USp(4)$, and just one real, or Hermitian, central charge can be implemented into the superalgebra.

The $USp(4)$ superalgebra is given as eqs. (2.1)–(2.8) with its R-symmetry group replaced by

[13] We call this theory the $SU(4)$ model.

[14] We call this theory the $USp(4)$ model.

$USp(4)$, namely[15],

$$
\begin{aligned}
\{Q_{i\alpha}, Q_{j\beta}\} &= C_{\alpha\beta}\Omega_{ij}Z, & \{\overline{Q}_{i\dot{\alpha}}, \overline{Q}_{j\dot{\beta}}\} &= C_{\dot{\alpha}\dot{\beta}}\Omega_{ij}Z, && (4.1)\\
\{Q_{i\alpha}, \overline{Q}_{j\dot{\beta}}\} &= 2\Omega_{ij}(\sigma^{\mu})_{\alpha\dot{\beta}}P_{\mu}, & [P_{\mu}, P_{\nu}] &= 0, && (4.2)\\
[Q_{i\alpha}, P_{\mu}] &= 0, & [\overline{Q}_{i\dot{\alpha}}, P_{\mu}] &= 0, && (4.3)\\
[J_{\mu\nu}, Q_{i\alpha}] &= \frac{i}{2}(\sigma_{\mu\nu})_{\alpha}{}^{\beta}Q_{i\beta}, & [J_{\mu\nu}, \overline{Q}_{i}{}^{\dot{\alpha}}] &= \frac{i}{2}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\overline{Q}_{i}{}^{\dot{\beta}}, && (4.4)\\
[J_{\mu\nu}, P_{\rho}] &= i(\eta_{\nu\rho}P_{\mu} - \eta_{\mu\rho}P_{\nu}), & [J_{\mu\nu}, J_{\rho\sigma}] &= i\left(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\nu\sigma}J_{\mu\rho}\right. && (4.5)\\
& & &\quad \left.- \eta_{\mu\rho}J_{\nu\sigma} + \eta_{\mu\sigma}J_{\nu\rho}\right), &&\\
[R^{a}, Q_{i\alpha}] &= (X^{a})_{i}{}^{j}Q_{j\alpha}, & [R^{a}, \overline{Q}_{i\dot{\alpha}}] &= (X^{a})_{i}{}^{j}\overline{Q}_{j\dot{\alpha}}, && (4.6)\\
[R^{a}, P_{\mu}] &= 0, & [R^{a}, J_{\mu\nu}] &= 0, && (4.7)\\
[Z, \mathrm{any}] &= 0, & & && (4.8)
\end{aligned}
$$

where supercharges $Q_{i\alpha}$ and $\overline{Q}^{i}{}_{\dot{\alpha}}$, related as

$$\overline{Q}^{i}{}_{\dot{\alpha}} := (Q_{i\alpha})^{\dagger}, \tag{4.9}$$

transform as Weyl spinors w.r.t. Lorentz transformations and as fundamental representations $\mathbf{4}$ and $\overline{\mathbf{4}}$, respectively, w.r.t. the internal $USp(4)$. These two internal symmetry representations are, however, equivalent and related so that indices are raised or lowered freely by the $USp(4)$ invariant metric Ω^{ij} and Ω_{ij} as in

$$Q^{i} = \Omega^{ij}Q_{j}, \quad Q_{i} = Q^{j}\Omega_{ji}, \qquad \Omega_{ij} + \Omega_{ji} = 0, \quad \Omega^{ik}\Omega_{jk} = \delta_{j}{}^{i}. \tag{4.10}$$

Thus we do not need to distinguish strictly the contravariant and covariant indices for $USp(4)$. We have denoted the $USp(4)$ generators by $R^{a} \in \mathfrak{usp}(4)$ and its representations by X^{a} which satisfy that

$$(X^{a})_{ij} = (X^{a})_{ji}, \quad (X^{a})_{ij} := (X^{a})_{i}{}^{k}\Omega_{kj}. \tag{4.11}$$

Through this section, we work on the theory in four-dimensional Minkowski spacetime, with $\eta_{\mu\nu} = (+,-,-,-)$. Note in passing that the central charge is in fact Hermitian

$$Z^{\dagger} = Z \tag{4.12}$$

as it has to be so since the Hermitian conjugate of one of the equation in (4.1) converts to the other.

4.3 The $USp(4)$ SYM

The $USp(4)$ super Yang-Mills theory is given as [11, 12]

$$
\begin{aligned}
S = \mathrm{tr}\int d^{4}x\,\Big(&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}V_{\mu}V^{\mu} + \frac{1}{2}D_{\mu}\phi_{ij}D^{\mu}\phi^{ij} + \frac{1}{2}H_{ij}H^{ij}\\
&- \frac{i}{4}\overline{\lambda}^{i}\overleftrightarrow{\not{D}}\lambda_{i} - \overline{\lambda}^{i}[\lambda^{j}, \phi_{ij}] + \frac{1}{4}[\phi_{ij}, \phi_{kl}][\phi^{ij}, \phi^{kl}]\Big),
\end{aligned}
\tag{4.13}
$$

where the field contents of the multiplet are arranged as

A_{μ} : vector, $4 - 1$ (gauge d.o.f.) $= 3$ components,

λ_{i} : $USp(4)$-Majorana spinors, $4 \times 4 = 16$ components,

[15] For notations on $USp(4)$ and $\mathfrak{usp}(4)$, see appendix A.4.

$$
\begin{aligned}
\phi_{ij}:&\ \text{scalars, 6 (antisymm. w.r.t. } i,j) -1\ (\Omega\text{-traceless}) = 5 \text{ components,}\\
V_\mu:&\ \text{pseudo vector, } 4-1 \text{ (gauge d.o.f.) } = 3 \text{ components,}\\
H_{ij}:&\ \text{auxiliary scalars, 5 components as in the scalar,}
\end{aligned}
$$

so that the d.o.f. of bosons and fermions are the same and are 16. The $USp(4)$-Majorana condition[16] is given as

$$
\lambda_{i\alpha} = \overline{\lambda}^{j\beta}(C^{-1})_{\beta\alpha}\Omega_{ji}, \tag{4.14}
$$

where C^{-1} is the charge conjugation matrix.

Supertransformations of these component fields are given as

$$
\delta A_\mu = i\overline{\zeta}^i(\gamma_\mu)\lambda_i, \tag{4.15}
$$

$$
\delta\phi_{ij} = -i\left(\overline{\zeta}_i\lambda_j - \overline{\zeta}_j\lambda_i + \frac{1}{2}\Omega_{ij}\overline{\zeta}^k\lambda_k\right), \tag{4.16}
$$

$$
\delta\lambda_i = -\frac{1}{2}\gamma_{\mu\nu}F^{\mu\nu}\zeta_i + 2\gamma^\mu D_\mu\phi_i{}^j\zeta_j + \gamma^5\gamma^\mu V_\mu\zeta_i + 2\gamma^5 H_i{}^j\zeta_j - 2i[\phi_{ik},\phi^{kj}]\zeta_j, \tag{4.17}
$$

$$
\begin{aligned}
\delta H_{ij} = {}& i(\overline{\zeta}_i\gamma^5\gamma^\mu D_\mu\lambda_j - \overline{\zeta}_j\gamma^5\gamma^\mu D_\mu\lambda_i + \frac{1}{2}\Omega_{ij}\overline{\zeta}^k\gamma^5\gamma^\mu D_\mu\lambda_k)\\
&- 2(\overline{\zeta}_i\gamma^5[\lambda^l,\phi_{jl}] - \overline{\zeta}_j\gamma^5[\lambda^l,\phi_{il}] + \frac{1}{2}\Omega_{ij}\overline{\zeta}^k\gamma^5[\lambda^l,\phi_{kl}]) + \overline{\zeta}^k\gamma^5[\lambda_k,\phi_{ij}]
\end{aligned} \tag{4.18}
$$

$$
\delta V_\mu = i\overline{\zeta}^i\gamma^5\gamma_{\mu\nu}D^\nu\lambda_i + 2\overline{\zeta}^i\gamma^5\gamma_\mu[\lambda^j,\phi_{ij}], \tag{4.19}
$$

and transformations generated by the central charge are

$$
\delta_z A_\mu = \omega V_\mu, \tag{4.20}
$$

$$
\delta_z\phi_{ij} = -\omega H_{ij}, \tag{4.21}
$$

$$
\delta_z\lambda_i = -\omega(\gamma^5\gamma^\mu D_\mu\lambda_i - 2i\gamma^5[\lambda^j,\phi_{ij}]), \tag{4.22}
$$

$$
\delta_z H_{ij} = \omega\left(-D^\mu D_\mu\phi_{ij} + i(\frac{1}{4}\Omega_{ij}\{\overline{\lambda}_k,\lambda^k\}) - \Big[[\phi_{kl},\phi^{kl}],\phi_{ij}\Big]\right), \tag{4.23}
$$

$$
\delta_z V_\mu = \omega\left(D_\nu F^\nu{}_\mu - \frac{1}{2}\{\overline{\lambda},\gamma_\mu\lambda\} + i[\phi^{ij},D_\mu\phi_{ij}]\right), \tag{4.24}
$$

where we have omitted the spinor indices. These supertransformations close off-shell, up to field dependent gauge transformations. The action (4.13) is invariant under these supertransformations as long as the additional constraint

$$
0 = D^\mu V_\mu + \frac{1}{2}\{\overline{\lambda}^i,\gamma^5\lambda_i\} - i[\phi^{ij},H_{ij}] \tag{4.25}
$$

is satisfies.

4.4 The $USp(4)$ superspace

The $USp(4)$ superspace is represented by the coordinates $(x^\mu,\theta^{i\alpha},\overline{\theta}^{i\dot{\alpha}},z)$ which are the conjugate to the generators $(P_\mu,Q_{i\alpha},\overline{Q}_{i\dot{\alpha}},Z)$. In other words, the superspace parameterizes, as local coordinate system, the manifold composed of the $USp(4)$ superalgebra. In this viewpoint, these generators of the superalgebra are represented as the adjoint representation on the manifold itself, and are expressed by the superspace coordinates as in

$$
P_\mu = i\partial_\mu, \tag{4.26}
$$

[16] See appendix B for the Majorana spinor representations.

$$Q_{i\alpha} = \frac{\partial}{\partial\theta^{i\alpha}} - i(\sigma^\mu)_{\alpha\dot\beta}\overline{\theta}_i{}^{\dot\beta}\partial_\mu - \frac{i}{2}\theta_{i\alpha}\partial_z, \tag{4.27}$$

$$\overline{Q}_{i\dot\alpha} = \frac{\partial}{\partial\overline{\theta}^{i\dot\alpha}} - i\theta_i{}^\beta(\sigma^\mu)_{\beta\dot\alpha}\partial_\mu - \frac{i}{2}\overline{\theta}_{i\dot\alpha}\partial_z, \tag{4.28}$$

$$Z = i\partial_z, \tag{4.29}$$

which satisfy that

$$\{Q_{i\alpha}, Q_{j\beta}\} = -C_{\alpha\beta}\Omega_{ij}Z, \qquad \{\overline{Q}_{i\dot\alpha}, \overline{Q}_{j\dot\beta}\} = -C_{\dot\alpha\dot\beta}\Omega_{ij}Z, \tag{4.30}$$

$$\{Q_{i\alpha}, \overline{Q}_{j\dot\beta}\} = -2\Omega_{ij}(\sigma^\mu)_{\alpha\dot\beta}P_\mu, \qquad [P_\mu, P_\nu] = 0, \tag{4.31}$$

$$[Q_{i\alpha}, P_\mu] = 0, \qquad [\overline{Q}_{i\dot\alpha}, P_\mu] = 0, \tag{4.32}$$

$$[Z, \text{any}] = 0. \tag{4.33}$$

Supercovariant derivatives, which all (anti-)commute with generators above are then given, on the superspace, as

$$D_\mu = \partial_\mu, \tag{4.34}$$

$$D_{i\alpha} = \frac{\partial}{\partial\theta^{i\alpha}} + i(\sigma^\mu)_{\alpha\dot\beta}\overline{\theta}_i{}^{\dot\beta}\partial_\mu + \frac{i}{2}\theta_{i\alpha}\partial_z, \tag{4.35}$$

$$\overline{D}_{i\dot\alpha} = \frac{\partial}{\partial\overline{\theta}^{i\dot\alpha}} + i\theta_i{}^\beta(\sigma^\mu)_{\beta\dot\alpha}\partial_\mu + \frac{i}{2}\overline{\theta}_{i\dot\alpha}\partial_z, \tag{4.36}$$

$$D_z = \partial_z, \tag{4.37}$$

which satisfy the following algebra:

$$\{D_{i\alpha}, D_{j\beta}\} = iC_{\alpha\beta}\Omega_{ij}D_z, \qquad \{\overline{D}_{i\dot\alpha}, \overline{D}_{j\dot\beta}\} = iC_{\dot\alpha\dot\beta}\Omega_{ij}D_z, \tag{4.38}$$

$$\{D_{i\alpha}, \overline{D}_{j\dot\beta}\} = 2i\Omega_{ij}(\sigma^\mu)_{\alpha\dot\beta}D_\mu, \qquad [D_\mu, D_\nu] = 0, \tag{4.39}$$

$$[D_{i\alpha}, D_\mu] = 0, \qquad [\overline{D}_{i\dot\alpha}, D_\mu] = 0, \tag{4.40}$$

$$[D_z, \text{any}] = 0, \tag{4.41}$$

$$\{Q_{i\alpha}, \text{any } D\} = 0, \qquad \{\overline{Q}_{i\dot\alpha}, \text{any } D\} = 0, \tag{4.42}$$

$$[P_\mu, \text{any } D] = 0, \qquad [Z, \text{any } D] = 0, \tag{4.43}$$

with the last four equations assuring the supercovariance of these derivatives.

4.5 Supersymmetric Gauge Theories on the $USp(4)$ Superspace

Let us now formulate a supersymmetric gauge theory on the $USp(4)$ superspace constructed in the preceding section. We follow the formulation using superconnections and supercurvatures defined on the superspace. This is technically the natural generalization of the formulation of ordinary gauge theory, i.e. Yang-Mills theory, using gauge connections and curvatures on the ordinary spacetime. We emphasize that this formulation has a great advantage among others since there the structure of the superalgebra can be manifestly implemented.

4.5.1 Superconnections

Supercovariant derivatives $D_I = (D_\mu, D_{i\alpha}, \overline{D}_{i\dot\alpha}, D_z)$ are further gauge covariantized by adding the superconnections Γ_I to define the gauge-supercovariant derivatives

$$\nabla_I := D_I - i\Gamma_I, \tag{4.44}$$

with gauge transformations

$$\nabla_I' = e^K \nabla e^{-K}, \quad \text{or} \quad \delta_K \nabla_I = [\nabla_I, K], \quad \delta_K \Gamma_I = i[\nabla_I, K], \tag{4.45}$$

where K is an anti-Hermitian gauge parameter superfield.

Here we note that by some suitable gauge transformations the lowest components w.r.t. θ, $\overline{\theta}$ of the fermionic superconnections $\Gamma_{i\alpha}$ and $\overline{\Gamma}_{i\dot{\alpha}}$ can be algebraically gauged away. On the other hand, the whole components of one of the five bosonic superconnections Γ_μ and Γ_z can also be gauged away as in the usual gauge theory. In fact, take the anti-Hermitian gauge parameter superfield K as, say,

$$K = \kappa(x,z) + \theta^{i\alpha}\left(i\Gamma_{i\alpha} - i[\kappa(x,z), \Gamma_{i\alpha}]\right) + \overline{\theta}^{i\dot{\alpha}}\left(i\overline{\Gamma}_{i\dot{\alpha}} - i[\kappa(x,z), \overline{\Gamma}_{i\dot{\alpha}}]\right) + K'(x,z,\theta,\overline{\theta}), \tag{4.46}$$

where K', which is also anti-Hermitian, contains no linear and zeroth order terms w.r.t. θ or $\overline{\theta}$ and the lowest parameter pure-imaginary field κ is determined as such that

$$\partial_z \kappa(x,z) = i[\Gamma_z(x, \theta=0, \overline{\theta}=0, z), \kappa(x,z)] + i\Gamma_z(x, \theta=0, \overline{\theta}=0, z). \tag{4.47}$$

Since this condition on κ is a first order differential equation, it has certainly a solution. Then it is easy to see

$$\Gamma'_{i\alpha}| = \Gamma_{i\alpha} + \delta\Gamma_{i\alpha}\Big| = 0, \quad \overline{\Gamma}'_{i\dot{\alpha}}| = \overline{\Gamma}_{i\dot{\alpha}} + \delta\overline{\Gamma}_{i\dot{\alpha}}\Big| = 0, \quad \Gamma'_z| = \Gamma_z + \delta\Gamma_z\Big| = 0, \tag{4.48}$$

where for any function on the superspace,

$$B| := B(x, \theta=0, \overline{\theta}=0, z). \tag{4.49}$$

We call such special gauge the Wess-Zumino gauge; namely in the Wess-Zumino gauge, we have

$$\Gamma_{i\alpha}| = \overline{\Gamma}_{i\dot{\alpha}}|\Gamma_z| = 0. \tag{4.50}$$

In the following computations, we usually take this gauge.

4.5.2 Supercurvatures

Supercurvatures F_{IJ} are then defined as in

$$[\nabla_I, \nabla_J]_\pm = -iF_{IJ}, \tag{4.51}$$

where the left hand side means anticommutators if both ∇_I and ∇_J are fermionic and denoted by the subscription $+$ while, otherwise, it means commutators and labeled by the subscription $-$. Gauge transformations on supercurvatures are read off from those on gauge-supercovariant derivatives as

$$\delta_K F_{IJ} = i[\delta_K \nabla_I, \nabla_J]_\pm + i[\nabla_I, \delta_K \nabla_J]_\pm = i[[\nabla_I, K], \nabla_J]_\pm + i[\nabla_I, [\nabla_J, K]]_\pm \tag{4.52}$$

$$= i[[\nabla_I, \nabla_J]_\pm, K] = [F_{IJ}, K]. \tag{4.53}$$

Thus supercurvatures are gauge covariant. They should therefore be the ingredients for constructing the super Yang-Mills theory. Also supercurvatures are supercovariant, i.e. they are superfields which transform correctly under a supertransformation. This fact can also be shown easily from the supercovariance of the superconnections.

Note for example

$$\{\nabla_{i\alpha}, \overline{\nabla}_{j\dot{\beta}}\}_\pm = 2\Omega_{ij}(\sigma^\mu)_{\alpha\dot{\beta}}\nabla_\mu - i\{D_{i\alpha}, \overline{\Gamma}_{j\dot{\beta}}\} - i\{\overline{D}_{j\dot{\beta}}, \Gamma_{i\alpha}\} - \{\Gamma_{i\alpha}, \overline{\Gamma}_{j\dot{\beta}}\} + 2i\Omega_{ij}(\sigma^\mu)_{\alpha\dot{\beta}}\Gamma_\mu. \tag{4.54}$$

The first term in the right hand side of this equation contains a gauge-supercovariant derivative itself. In literature, such terms are sometimes called supertorsions and distinguished from supercurvatures. It is of course merely the difference of terminology, and here, we call, including such terms, the right hand side of eq. (4.51) supercurvatures.

4.5.3 Constraints on Supercurvatures

In an ordinary Yang-Mills theory, once a gauge covariant curvature are defined, one can construct the Lagrangian (action) as $\mathcal{L} \sim -\operatorname{tr} F_{\mu\nu}F^{\mu\nu}$. In supersymmetric theories, however, superfields have large degrees of freedom so highly reducible, containing possibly many extra component fields. Thus to formulate the correct theory one would like to construct, one has to impose some appropriate constraints on superfields and restricts them covariantly to the symmetry of the system. In our case we need some suitable gauge and supercovariant constraints on the supercurvatures we defined above.

To obtain appropriate constraints, let us start by writing the supercurvatures constructed from two fermionic derivatives as

$$\{\nabla_{i\alpha}, \nabla_{j\beta}\} = i\Omega_{ij}C_{\alpha\beta}\nabla_z - iF_{ij\alpha\beta}, \tag{4.55}$$
$$\{\nabla_{i\alpha}, \overline{\nabla}_{j\dot{\beta}}\} = i\Omega_{ij}(\sigma^\mu)_{\alpha\dot{\beta}}\nabla_\mu - iF_{ij\alpha\dot{\beta}}, \tag{4.56}$$

where[17]

$$F_{ij\alpha\beta} = F_{ji\beta\alpha} \tag{4.57}$$

since the left hand side of eq. (4.55) is symmetric under the exchange $(i,\alpha) \leftrightarrow (j,\beta)$. Then it can be decomposed into symmetric and antisymmetric parts as in

$$F_{ij\alpha\beta} = W^{\mathrm{a}}_{ij\alpha\beta} + W^{\mathrm{s}}_{ij\alpha\beta}, \tag{4.58}$$

where

$$W^{\mathrm{a}}_{ij\alpha\beta} + W^{\mathrm{a}}_{ji\alpha\beta} = W^{\mathrm{a}}_{ij\alpha\beta} + W^{\mathrm{a}}_{ij\beta\alpha} = 0, \tag{4.59}$$
$$W^{\mathrm{s}}_{ij\alpha\beta} = W^{\mathrm{s}}_{ji\alpha\beta} = W^{\mathrm{s}}_{ij\beta\alpha}. \tag{4.60}$$

The antisymmetric part is trivially written as

$$W^{\mathrm{a}}_{ij\alpha\beta} = C_{\alpha\beta}W^{\mathrm{a}}_{ij}, \quad W^{\mathrm{a}}_{ij} + W^{\mathrm{a}}_{ji} = 0. \tag{4.61}$$

Then further we can decompose the antisymmetric factor into the parts which is proportional to the invariant metric Ω_{ij} and which does not contain Ω_{ij} as in

$$W^{\mathrm{a}}_{ij} = W_{ij} + \Omega_{ij}W, \quad \text{where} \quad \Omega^{ij}W_{ij} = 0 \quad (\Omega\text{-traceless}). \tag{4.62}$$

Similarly, the symmetric part can be represented as

$$W^{\mathrm{s}}_{ij\alpha\beta} = \frac{1}{2}\sum_a (\sigma_{\mu\nu})_{\alpha\beta}(X^a)_{ij}W^{\mu\nu a}, \tag{4.63}$$
$$\text{where} \quad (X^a)_{ij} \in \mathfrak{usp}(4), \tag{4.64}$$

since $(\sigma_{\mu\nu})_{\alpha\beta} = (\sigma_{\mu\nu})_{\beta\alpha}$ and $X_{ij} = X_{ji}$ for $(X_{ij}) \in \mathfrak{usp}(4)$.

[17] We have rescale the gauge-supercovariant derivatives as

$$\nabla_\alpha \to \sqrt{2}\nabla_\alpha, \quad \overline{\nabla}_{\dot{\alpha}} \to \sqrt{2}\overline{\nabla}_{\dot{\alpha}}, \quad \nabla_z \to 2\nabla_z$$

just for a conventional reason.

Spins and Constraints We now concentrate on the spins contained in the supercurvatures. Spins (helicities) are of course eigenvalues of the Lorentz generators $J_{\mu\nu}$. Here we notice

$$[S_i^\pm, S_j^\pm] = i\epsilon_{ijk}S^{\pm k}, \quad [S_i^\pm, S_j^\mp] = 0, \qquad S_i^\pm := \frac{1}{2}\left(-\frac{1}{2}\epsilon_{ijk}J^{jk} \mp J^4{}_i\right) = -\frac{1}{2}\epsilon_{ijk}(J^\pm)^{jk}, \tag{4.65}$$

where $(J^\pm)_{\mu\nu}$ are the selfdual and anti-selfdual parts. We thus recognize $S_i^\pm$ as the spin generators. The third component of spins (helicities) are then

$$J := (S^+)^3 + (S^-)^3 = J^{12}. \tag{4.66}$$

Then using

$$[J_{\mu\nu}, Q_{i\alpha}] = \frac{i}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_{i\beta}, \quad [J_{\mu\nu}, \overline{Q}_{i\dot\alpha}] = \frac{i}{2}(\bar\sigma_{\mu\nu})^{\dot\beta\dot\alpha}\overline{Q}_{i\dot\beta}, \tag{4.67}$$

we can easily compute spins of the superfields $F_{ij\alpha\beta}$ and $F_{ij\alpha\dot\beta}$, as in

$$[J, F_{ij1\dot1}] = +F_{ij1\dot1}, \qquad [J, F_{ij2\dot2}] = -F_{ij2\dot2}, \tag{4.68}$$

$$[J, F_{ij1\dot2}] = [J, F_{ij2\dot1}] = 0. \tag{4.69}$$

and

$$[J, W^{\rm s}_{ij11}] = +W^{\rm s}_{ij11}, \qquad [J, W^{\rm s}_{ij22}] = -W^{\rm s}_{ij22}, \tag{4.70}$$

$$[J, W^{\rm s}_{ij12}] = [J, W^{\rm s}_{ij21}] = 0. \tag{4.71}$$

Thus we find that $F_{ij\alpha\dot\beta}$ and $W^{\rm s}$ have spin 1. As will be clear later, the multiplet in our formulation is determined by successively multiplying ∇_α, $\overline{\nabla}_{\dot\alpha}$ as "creation operators" on supercurvatures as the basic superfields. In other words, component fields are created as the form

$$\nabla_{I_1}\cdots\nabla_{I_r}F, \tag{4.72}$$

where F is a nonzero supercurvature in our system. Since $F_{ij\alpha\dot\beta}$ and $W^{\rm s}$ have spins 1 themselves, the created fields from such superfields can obviously contain spins higher than 1, which are not allowed in the ($USp(4)$) super Yang-Mills multiplet. We therefore drop such superfields:

$$F_{ij\alpha\dot\beta} = W^{\rm s} = 0. \tag{4.73}$$

Hence we consider the constraints

$$\{\nabla_{i\alpha}, \nabla_{j\beta}\} = i\Omega_{ij}C_{\alpha\beta}\nabla_z - i\Omega_{ij}C_{\alpha\beta}W - iC_{\alpha\beta}W_{ij}, \tag{4.74}$$

$$\{\nabla_{i\alpha}, \overline{\nabla}_{j\dot\beta}\} = i\Omega_{ij}(\sigma^\mu)_{\alpha\dot\beta}\nabla_\mu, \tag{4.75}$$

where $\Omega_{ij}C_{\alpha\beta}W$ at the right hand side of the first equation can be absorbed into $\Omega_{ij}C_{\alpha\beta}\Gamma_z$ by a trivial redefinition of the superconnection Γ_z.

Reality Condition Finally let us consider the reality of W_{ij} (and, strictly, W, in order to be absorbed in Γ_z as noted above). Since $USp(4)$ is real in the sense of representations, we can impose the condition, consistently to the internal symmetry,

$$(W_{ij})^* \equiv \overline{W}^{ij} = W^{ij}, \quad W^* = W. \tag{4.76}$$

where

$$W^{ij} := \Omega^{ik}\Omega^{jl}W_{kl}. \tag{4.77}$$

Since we are looking for the irreducible (or minimal) multiplet, we must impose all we can impose without breaking any nontrivial structure of the constraints. More practically, the scalar fields contained in the $USp(4)$ model are real, so that we do not need extra complex d.o.f. of the superfields W_{ij} (and W). We therefore adopt these reality constraints. We emphasize here that in the earlier attempt to construct the $SU(4)$ model in [10], the nontrivial complex structure of $SU(4)$ prevented from imposing any such reality constraint without forcing the superalgebra close only on-shell.

Thus our constraints in the final form are

$$\{\nabla_{i\alpha}, \nabla_{j\beta}\} = i\Omega_{ij}C_{\alpha\beta}\nabla_z - iC_{\alpha\beta}W_{ij}, \qquad \{\overline{\nabla}_{i\dot{\alpha}}, \overline{\nabla}_{j\dot{\beta}}\} = i\Omega_{ij}C_{\dot{\alpha}\dot{\beta}}\nabla_z + iC_{\dot{\alpha}\dot{\beta}}W_{ij}, \tag{4.78}$$

$$\{\nabla_{i\alpha}, \overline{\nabla}_{j\dot{\beta}}\} = i\Omega_{ij}(\sigma^\mu)_{\alpha\dot{\beta}}\nabla_\mu, \tag{4.79}$$

$$[\nabla_{i\alpha}, \nabla_\mu] = -iF_{i\alpha\mu}, \qquad [\overline{\nabla}_{i\dot{\alpha}}, \nabla_\mu] = +i\overline{F}_{i\dot{\alpha}\mu}, \tag{4.80}$$

$$[\nabla_{i\alpha}, \nabla_z] = -iG_{i\alpha}, \qquad [\overline{\nabla}_{i\dot{\alpha}}, \nabla_z] = +i\overline{G}_{i\dot{\alpha}}, \tag{4.81}$$

$$[\nabla_\mu, \nabla_z] = -ig_\mu \qquad [\nabla_\mu, \nabla_\nu] = -iF_{\mu\nu}, \tag{4.82}$$

where

$$\Omega^{ij}W_{ij} = 0, \quad (W_{ij})^* = W^{ij}. \tag{4.83}$$

4.6 Solving the Constraints

Let us then move on to solving the constraints (4.78)–(4.82) derived above.

Specifically, we will solve them in the following way. First, applying the constraints into the Bianchi identities

$$[\nabla_A, [\nabla_B, \nabla_C\}\} \pm [\nabla_B, [\nabla_C, \nabla_A\}\} \pm [\nabla_C, [\nabla_A, \nabla_B\}\} = 0, \tag{4.84}$$

we obtain various relations among the supercurvatures and their derivatives. Then we will find each higher derivatives of the supercurvatures, particularly of the superfield W_{ij}, can be expressed only by some combinations of all the other supercurvatures, including themselves, and their lower derivatives. In other words, we can compute each higher derivatives of the supercurvatures self-consistently using the other supercurvatures in our system. Once each derivatives of supercurvatures are computed self-consistently, componentwise expansions of the supercurvatures can be fulfilled, so that the supercurvatures are determined completely in a self-consistent manner. In this sense we say we solve the constraints. We will then identify the set of component fields, which are necessary and sufficient to compute all the supercurvatures componentwisely, as the independent fields in our system.

4.6.1 Bianchi Identities

We now go through with the process described above. Since the computations are lengthy and cumbersome, here we only list the results in the following[18].

Three Fermionic Derivatives First we list the results of applying Bianchi identities for three fermionic derivatives, say $(\nabla_\alpha, \nabla_\beta, \nabla_\gamma)$.

$$[\nabla_{j\alpha}, W^j{}_i] = 5iG_{i\alpha}, \qquad [\overline{\nabla}_{j\dot{\alpha}}, W^j{}_i] = 5i\overline{G}_{i\dot{\alpha}}, \tag{4.85}$$

$$G_{i\alpha} = -\frac{1}{4}(\sigma^\mu C)_\alpha{}^{\dot{\beta}}\overline{F}_{i\dot{\beta}\mu}, \qquad \overline{G}_{i\dot{\alpha}} = -\frac{1}{4}(C\sigma^\mu)^\beta_{\dot{\alpha}}F_{i\beta\mu}, \tag{4.86}$$

$$[\nabla_{i\alpha}, W_{jk}] = 2i\Omega_{i[j}G_{k]\alpha} + i\Omega_{jk}G_{i\alpha}, \qquad [\overline{\nabla}_{i\dot{\alpha}}, W_{jk}] = 2i\Omega_{i[j}\overline{G}_{k]\dot{\alpha}} + i\Omega_{jk}\overline{G}_{i\dot{\alpha}}, \tag{4.87}$$

$$F_{i\alpha\mu} = (\bar{\sigma}_\mu C)^{\dot{\gamma}}{}_\alpha\overline{G}_{i\dot{\gamma}}, \qquad \overline{F}_{i\dot{\alpha}} = (C\bar{\sigma}_\mu)_{\dot{\alpha}}{}^\beta G_{i\beta}. \tag{4.88}$$

[18] See appendix C for the detail of those computations.

Four Fermionic Derivatives

$$\{\nabla_{i\alpha}, G_{j\beta}\} = -\frac{i}{4}\Omega_{ij}(\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu} - \frac{1}{2}C_{\alpha\beta}[\nabla_z, W_{ij}] - \frac{1}{4}C_{\alpha\beta}[W_{ik}, W^k{}_j], \tag{4.89}$$

$$\{\overline{\nabla}_{i\dot\alpha}, \overline{G}_{j\dot\beta}\} = -\frac{i}{4}\Omega_{ij}(\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta}F_{\mu\nu} - \frac{1}{2}C_{\dot\alpha\dot\beta}[\nabla_z, W_{ij}] + \frac{1}{4}C_{\dot\alpha\dot\beta}[W_{ik}, W^k{}_j], \tag{4.90}$$

$$\{\overline{\nabla}_{i\dot\alpha}, G_{j\beta}\} = -\frac{1}{2}(\sigma^\mu)_{\beta\dot\alpha}\Big(i\Omega_{ij}g_\mu - [\nabla_\mu, W_{ij}]\Big), \tag{4.91}$$

$$\{\nabla_{i\alpha}, \overline{G}_{j\dot\beta}\} = -\frac{1}{2}(\sigma^\mu)_{\alpha\dot\beta}\Big(i\Omega_{ij}g_\mu + [\nabla_\mu, W_{ij}]\Big). \tag{4.92}$$

Five Fermionic Derivatives

$$[\nabla_{i\alpha}, g_\mu] = -(\sigma^{\mu\nu})_\alpha{}^\beta[\nabla^\nu, G_{i\beta}] + (\bar\sigma_\mu C)^{\dot\beta}{}_\alpha[\overline{G}^j{}_{\dot\beta}, W_{ij}], \tag{4.93}$$

$$[\overline{\nabla}_{i\dot\alpha}, g_\mu] = -(\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\alpha}[\nabla^\nu, \overline{G}_{i\dot\beta}] + (C\bar\sigma_\mu)_{\dot\alpha}{}^\beta[G^j{}_\beta, W_{ij}], \tag{4.94}$$

$$[\nabla_z, G_{i\alpha}] = -(\sigma^\mu C)_\alpha{}^{\dot\alpha}[\nabla_\mu, \overline{G}_{i\dot\alpha}] + [G^j{}_\alpha, W_{ij}], \tag{4.95}$$

$$[\nabla_z, \overline{G}_{i\dot\alpha}] = -(C\sigma^\mu)^\beta{}_{\dot\alpha}[\nabla_\mu, G_{i\beta}] - [\overline{G}^j{}_{\dot\alpha}, W_{ij}], \tag{4.96}$$

$$\begin{aligned}[\nabla_{i\alpha}, [\nabla_z, W_{jk}]] = &-i(\sigma^\mu C)_\alpha{}^{\dot\alpha}\Big(\Omega_{jk}[\nabla_\mu, \overline{G}_{i\dot\alpha}] + 2\Omega_{i[j}[\nabla_\mu, \overline{G}_{k]\dot\alpha}]\Big)\\ &+ i\Big(\Omega_{jk}[G^l{}_\alpha, W_{il}] + 2\Omega_{i[j}[G^l{}_\alpha, W_{k]l}]\Big)\\ &- i[G_{i\alpha}, W_{jk}],\end{aligned} \tag{4.97}$$

$$\begin{aligned}[\overline{\nabla}_{i\dot\alpha}, [\nabla_z, W_{jk}]] = &-i(C\sigma^\mu)^\beta{}_{\dot\alpha}\Big(\Omega_{jk}[\nabla_\mu, G_{i\beta}] + 2\Omega_{i[j}[\nabla_\mu, G_{k]\beta}]\Big)\\ &- i\Big(\Omega_{jk}[\overline{G}^l{}_{\dot\alpha}, W_{il}] + 2\Omega_{i[j}[\overline{G}^l{}_{\dot\alpha}, W_{k]l}]\Big)\\ &+ i[\overline{G}_{i\dot\alpha}, W_{jk}].\end{aligned} \tag{4.98}$$

Six Fermionic Derivatives

$$\begin{aligned}[\nabla_z, [\nabla_z, W_{jk}]] = [\nabla_\mu, &[\nabla^\mu, W_{jk}]]\\ &+ i(\Omega_{jk}\{G_{i\alpha}, G^{i\alpha}\} - 4\{G_{j\alpha}, G_k{}^\alpha\})\\ &- i(\Omega_{jk}\{\overline{G}_{i\dot\alpha}, \overline{G}^{i\dot\alpha}\} - 4\{\overline{G}_{j\dot\alpha}, \overline{G}_k{}^{\dot\alpha}\})\\ &+ \frac{1}{4}[W_{[j|l}, [W_{k]m}, W^{ml}]],\end{aligned} \tag{4.99}$$

$$[\nabla_z, g_\mu] = [\nabla^\nu, F_{\mu\nu}] - 2(\bar\sigma^{\dot\alpha\alpha})\{G_{i\alpha}, \overline{G}^i{}_{\dot\alpha}\} + \frac{i}{4}[W^{ij}, [\nabla_\mu, W_{ij}]], \tag{4.100}$$

$$[\nabla^\mu, g_\mu] = -\{G_{i\alpha}, G^{i\alpha}\} - \{\overline{G}_{i\dot\alpha}, \overline{G}^{i\dot\alpha}\} + \frac{i}{4}[W^{ij}, [\nabla_z, W_{ij}]]. \tag{4.101}$$

Note each first fermionic derivative of all the superfields in the theory has been consistently computed in terms of only these superfields. In this sense, these computations close in the set of superfields at our disposal. Especially no additional (super)field can be excited by applying fermionic derivatives in any way. In other words, we can compute any higher derivatives of these superfields. Such computations including further fermionic derivatives are necessary to obtain an invariant action.

4.6.2 Definition of Component Fields

According to the computations above, we find the independent component fields, or essentially the independent superfields, which should correspond to the following degrees of freedom:

$$\phi_{ij} := W_{ij}|, \tag{4.102}$$

$$\lambda_{i\alpha} := 2iG_{i\alpha}| = \frac{2}{5}[\nabla_{j\alpha}, W^j{}_i]\big| = -2[\nabla_{i\alpha}, \nabla_z]\big|, \tag{4.103}$$

$$\overline{\lambda}_{i\dot{\alpha}} := -2i\overline{G}_{i\dot{\alpha}}| = -\frac{2}{5}[\overline{\nabla}_{j\dot{\alpha}}, W^j{}_i]\big| = -2[\overline{\nabla}_{i\dot{\alpha}}, \nabla_z]\big|, \tag{4.104}$$

$$H_{ij} := -i[\nabla_z, W_{ij}]| \tag{4.105}$$

$$= \frac{1}{10}\{\nabla_{[i\alpha}, [\nabla_k{}^{\alpha}, W^k{}_{j]}]\}\big| = \frac{i}{2}\{\nabla_{[i\alpha}, G_{j]}{}^{\alpha}\}\big| \tag{4.106}$$

$$= \frac{1}{10}\{\overline{\nabla}_{[i\dot{\alpha}}, [\overline{\nabla}_k{}^{\dot{\alpha}}, W^k{}_{j]}]\}\big| = \frac{i}{2}\{\overline{\nabla}_{[i\dot{\alpha}}, \overline{G}_{j]}{}^{\dot{\alpha}}\}\big|, \tag{4.107}$$

$$V_\mu := ig_\mu| = -[\nabla_\mu, \nabla_z]\big| \tag{4.108}$$

$$= \frac{i}{20}(\bar{\sigma}_\mu)^{\dot{\beta}\beta}\{\nabla_{i\beta}, [\overline{\nabla}_{j\dot{\beta}}, W^{ji}]\}\big| \tag{4.109}$$

$$= \frac{i}{20}(\bar{\sigma}_\mu)^{\dot{\beta}\beta}\{\overline{\nabla}_{i\dot{\beta}}, [\nabla_{j\beta}, W^{ji}]\}\big|, \tag{4.110}$$

$$A_\mu := i\nabla_\mu|, \tag{4.111}$$

$$\Bigg(F_{\mu\nu} := i[\nabla_\mu, \nabla_\nu]\big|$$

$$= \frac{i}{8}\Big((\sigma_{\mu\nu})^{\alpha\beta}\{\nabla_{i\alpha}, G^i{}_\beta\} + (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}}\{\overline{\nabla}_{i\dot{\alpha}}, \overline{G}^i{}_{\dot{\beta}}\}\Big)\Big|$$

$$= \frac{1}{40}\Big((\sigma_{\mu\nu})^{\alpha\beta}\{\nabla_{i\alpha}, [\nabla_{j\beta}, W^{ji}]\} + (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}}\{\overline{\nabla}_{i\dot{\alpha}}, [\overline{\nabla}_{j\dot{\beta}}, W^{ji}]\}\Big)\Big| \quad \Bigg), \tag{4.112}$$

where | denotes to take the lowest components. Note these field contents are exactly the same as in the $USp(4)$ model in [10]. In particular, the highest spin is 1, successfully prohibiting the higher spins.

4.6.3 Supertransformations

Let us now derive the supertransformations on these component fields. Actually we find that these are the same as in eq. (4.15)–(4.19), up to some trivial constant factors and the difference of the spinor representations, as they should be. Similarly the central charge transformations are also rederived.

First we recall how those transformations are computed in our formulation. Let B be a general superfield and $b := B|$ its lowest component field. Supertransformation of the component field b is defined to be

$$\delta b = \delta(B|) := (\delta B)\Big| = [\xi^{i\alpha}Q_{i\alpha} + \overline{\xi}^{i\dot{\alpha}}\overline{Q}_{i\dot{\alpha}}, B]\Big| = [\xi^{i\alpha}D_{i\alpha} + \overline{\xi}^{i\dot{\alpha}}\overline{D}_{i\dot{\alpha}}, B]\Big|, \tag{4.113}$$

where supercharges are represented as differential operators on the superspace. The last equality holds since the computation is for the lowest terms; the difference of $Q_{i\alpha}$, $\overline{Q}_{i\dot{\alpha}}$ and $D_{i\alpha}$, $\overline{D}_{i\dot{\alpha}}$ contains only linear terms w.r.t. θ, $\overline{\theta}$, so that it vanishes when taking the lowest contributions (unless the superfield B contains fermionic derivatives, which is the case in our computations below.). In the following computations, we always take the Wess-Zumino gauge, where the fermionic superconnections (and the central charge superconnection as well) have no lowest terms. Then we find

$$\delta b = \xi^{i\alpha}[D_{i\alpha}, B]_\pm + \overline{\xi}^{i\dot{\alpha}}[\overline{D}_{i\dot{\alpha}}, B]_\pm\Big| = \xi^{i\alpha}[\nabla_{i\alpha}, B]_\pm + \overline{\xi}^{i\dot{\alpha}}[\overline{\nabla}_{i\dot{\alpha}}, B]_\pm\Big|. \tag{4.114}$$

Thus supertransformations of $b = B|$ can be readily computed from the fermionic first derivatives of superfields $[\nabla_{i\alpha}, B]$ and $[\overline{\nabla}_{i\dot{\alpha}}, B]$. In our case, each independent component field is defined as a lowest component of the superfield in the system whose fermionic first derivatives have

been computed completely by Bianchi identities as shown in the preceding section. We can therefore derive any supertransformations of each component field in our formulation. Similarly, the central charge transformations of the field $b = B|$ are computed as

$$\delta_z b = \delta_z(B|) := (\delta_z B)\Big| = [i\omega D_z, B]\Big| = i\omega[\nabla_z, B]\Big| \qquad \text{(due to the WZ gauge).} \tag{4.115}$$

We then list below the explicit results.

$$\begin{aligned}
\delta\phi_{ij} &= \delta W_{ij}\Big| \\
&= \xi^{k\alpha}[\nabla_{k\alpha}, W_{ij}] + \overline{\xi}^{k\dot{\alpha}}[\overline{\nabla}_{k\dot{\alpha}}, W_{ij}]\Big| \\
&= \left(\xi_{[i}{}^{\alpha}\lambda_{j]\alpha} + \frac{1}{2}\Omega_{ij}\xi^{k\alpha}\lambda_{k\alpha}\right) - \left(\overline{\xi}_{[i}{}^{\dot{\alpha}}\overline{\lambda}_{j]\dot{\alpha}} + \frac{1}{2}\Omega_{ij}\overline{\xi}^{k\dot{\alpha}}\overline{\lambda}_{k\dot{\alpha}}\right), \qquad (4.116) \\
\delta\lambda_{i\alpha} &= 2i\left(\xi^{k\beta}\{\nabla_{k\beta}, G_{i\alpha}\} + \overline{\xi}^{k\dot{\beta}}\{\overline{\nabla}_{k\dot{\beta}}, G_{i\alpha}\}\right)\Big| \\
&= \frac{1}{2}\xi_i{}^{\beta}(\sigma^{\mu\nu})_{\beta\alpha}F_{\mu\nu} \\
&\quad + i(\sigma^{\mu})_{\alpha\dot{\beta}}\overline{\xi}^{k\dot{\beta}}[\nabla_\mu, \phi_{ki}] - i(\sigma^{\mu})_{\alpha\dot{\beta}}\overline{\xi}_i{}^{\dot{\beta}}V_\mu + i\xi^k{}_{\alpha}H_{ki} - \frac{i}{2}\xi^k{}_{\alpha}[\phi_{kl}, \phi^l{}_i], \qquad (4.117) \\
\delta\overline{\lambda}_{i\dot{\alpha}} &= -2i\left(\xi^{k\beta}\{\nabla_{k\beta}, \overline{G}_{i\dot{\alpha}}\} + \overline{\xi}^{k\dot{\beta}}\{\overline{\nabla}_{k\dot{\beta}}, \overline{G}_{i\dot{\alpha}}\}\right)\Big| \\
&= -\frac{1}{2}\overline{\xi}_i{}^{\dot{\beta}}(\bar{\sigma}^{\mu\nu})_{\beta\alpha}F_{\mu\nu} \\
&\quad + i(\sigma^{\mu})_{\beta\dot{\alpha}}\xi^{k\beta}[\nabla_\mu, \phi_{ki}] + i(\sigma^{\mu})_{\beta\dot{\alpha}}\xi_i{}^{\beta}V_\mu + i\overline{\xi}^k{}_{\dot{\alpha}}H_{ki} - \frac{i}{2}\overline{\xi}^k{}_{\dot{\alpha}}[\phi_{kl}, \phi^l{}_i], \qquad (4.118) \\
\delta H_{ij} &= -i\left(\xi^{k\beta}[\nabla_{k\beta}, [\nabla_z, W_{ij}]] + \overline{\xi}^{k\dot{\beta}}[\overline{\nabla}_{k\dot{\beta}}, [\nabla_z, W_{ij}]]\right)\Big| \\
&= -i\left(\xi_{[i}{}^{\beta}(\sigma^{\mu})_{\beta\dot{\beta}}[\nabla_\mu, \overline{\lambda}_{j]}{}^{\dot{\beta}}] + \frac{1}{2}\Omega_{ij}\xi^{k\beta}(\sigma_\mu)_{\beta\dot{\beta}}[\nabla_\mu, \overline{\lambda}_k{}^{\dot{\beta}}]\right) \\
&\quad - i\left(\overline{\xi}_{[i\dot{\beta}}(\bar{\sigma}^{\mu})^{\dot{\beta}\beta}[\nabla_\mu, \lambda_{j]\beta}] + \frac{1}{2}\Omega_{ij}\overline{\xi}^k{}_{\dot{\beta}}(\bar{\sigma}_\mu)^{\dot{\beta}\beta}[\nabla_\mu, \lambda_{k\beta}]\right) \\
&\quad - i\left(\xi_{[i}{}^{\beta}[\lambda^l{}_{\beta}, \phi_{j]l}] + \frac{1}{2}\Omega_{ij}\xi^{k\beta}[\lambda^l{}_{\beta}, \phi_{kl}]\right) \\
&\quad - i\left(\overline{\xi}_{[i}{}^{\dot{\beta}}[\overline{\lambda}^l{}_{\dot{\beta}}, \phi_{j]l}] + \frac{1}{2}\Omega_{ij}\overline{\xi}^{k\dot{\beta}}[\overline{\lambda}^l{}_{\dot{\beta}}, \phi_{kl}]\right) \\
&\quad + \frac{i}{2}\xi^{k\beta}[\lambda_{k\beta}, \phi_{ij}] + \frac{i}{2}\overline{\xi}^{k\dot{\beta}}[\overline{\lambda}_{k\dot{\beta}}, \phi_{ij}], \qquad (4.119) \\
\delta V_\mu &= i\xi^{i\alpha}[\nabla_{i\alpha}, g_\mu] + i\overline{\xi}^{i\dot{\alpha}}[\overline{\nabla}_{i\dot{\alpha}}, g_\mu]\Big| \\
&= -\frac{1}{2}\xi^{i\alpha}(\sigma_{\mu\nu})_{\alpha}{}^{\beta}[\nabla^\nu, \lambda_{i\beta}] + \frac{1}{2}\overline{\xi}^i{}_{\dot{\alpha}}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}[\nabla^\nu, \overline{\nabla}_i{}^{\dot{\beta}}] \\
&\quad + \frac{1}{2}\xi^{i\alpha}(\sigma_\mu)_{\alpha\dot{\alpha}}[\overline{\lambda}^{j\dot{\alpha}}, \phi_{ij}] + \frac{1}{2}\overline{\xi}^i{}_{\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}[\lambda^j{}_{\alpha}, \phi_{ij}], \qquad (4.120) \\
\delta A_\mu &= i\left(\xi^{i\alpha}[\nabla_{i\alpha}, \nabla_\mu] + \overline{\xi}^{i\dot{\alpha}}[\overline{\nabla}_{i\dot{\alpha}}, \nabla_\mu]\right)\Big| \\
&= -\frac{i}{2}\xi^{i\alpha}(\sigma_\mu)_{\alpha\dot{\alpha}}\overline{\lambda}_i{}^{\dot{\alpha}} + \frac{i}{2}\overline{\xi}^i{}_{\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}\lambda_{i\alpha}. \qquad (4.121)
\end{aligned}$$

Similarly, the central charge transformations are give as follows:

$$\begin{aligned}
\delta_z\phi_{ij} &= i\omega[\nabla_z, W_{ij}]\Big| \\
&= -\omega H_{ij}, \qquad (4.122)
\end{aligned}$$

$$\begin{aligned}
\delta_z\lambda_{i\alpha} &= -2\omega[\nabla_z, G_{i\alpha}]\Big| \\
&= i\omega\left((\sigma^\mu)_{\alpha\dot\alpha}[\nabla_\mu, \overline{\lambda}_i{}^{\dot\alpha}] + [\lambda^j{}_\alpha, \phi_{ij}]\right), && (4.123)\\
\delta_z\overline{\lambda}_{i\dot\alpha} &= -2\omega[\nabla_z, G_{i\alpha}]\Big| \\
&= -i\omega\left((C\bar\sigma^\mu)_{\dot\alpha}{}^\alpha[\nabla_\mu, \lambda_{i\alpha}] + [\overline{\lambda}^j{}_{\dot\alpha}, \phi_{ij}]\right), && (4.124)\\
\delta_z H_{ij} &= \omega[\nabla_z, [\nabla_z, W_{ij}]]\Big| \\
&= \omega\Bigg([\nabla_\mu, [\nabla^\mu, \phi_{ij}]] - \frac{1}{4}[\phi_{[i|k}, [\phi_{j]l}, \phi^{kl}]] \\
&\qquad - i\left(\frac{1}{4}\Omega_{ij}\{\lambda_{k\gamma}, \lambda^{k\gamma}\} - \{\lambda_{i\gamma}, \lambda_j{}^\gamma\}\right) \\
&\qquad + i\left(\frac{1}{4}\Omega_{ij}\{\overline{\lambda}_{k\dot\gamma}, \overline{\lambda}^{k\dot\gamma}\} - \{\overline{\lambda}_{i\dot\gamma}, \overline{\lambda}_j{}^{\dot\gamma}\}\right)\Bigg), && (4.125)\\
\delta_z V_\mu &= -\omega[\nabla_z, g_\mu]\Big| \\
&= \omega\left([\nabla_\nu, F^\nu{}_\mu] - \frac{1}{2}(\bar\sigma)^{\dot\alpha\alpha}\{\overline{\lambda}_{i\dot\alpha}, \lambda^i{}_\alpha\} + \frac{1}{4}[\phi^{ij}, [\nabla_\mu, \phi_{ij}]]\right), && (4.126)\\
\delta_z A_\mu &= -\omega[\nabla_z, \nabla_\mu]\Big| \\
&= -\omega V_\mu. && (4.127)
\end{aligned}$$

4.6.4 Off-Shell Closure

Finally we show that the supertransformations derived above close off-shell. Since the supertransformations are essentially intrinsic to our superspace formulation and derived automatically, the off-shell closure of those supertransformations shows, in a sense, the superspace has been completely and successfully constructed for the $USp(4)$ model.

Let B be one of the superfields in our case and $b := B|$ be the lowest component as before. Recall first the off-shell closure is generally represented as

$$\begin{aligned}
(\delta_1\delta_2 - \delta_2\delta_1)b = &-i\left((\xi_1)_{i\alpha}(\xi_2)^{i\alpha} + (\overline{\xi}_1)_{i\dot\alpha}(\overline{\xi}_2)^{i\dot\alpha}\right)[\partial_z, b] \\
&- i\left((\xi_1)_i{}^\alpha(\sigma^\mu)_{\alpha\dot\alpha}(\overline{\xi}_2)^{i\dot\alpha} - (\xi_2)_i{}^\alpha(\sigma^\mu)_{\alpha\dot\alpha}(\overline{\xi}_1)^{i\dot\alpha}\right)[\partial_\mu, b]. && (4.128)
\end{aligned}$$

Instead of showing this by using the explicit formulae (4.116)–(4.121), we presents here more general procedure in the WZ gauge:

$$\begin{aligned}
\delta b &= \xi^{i\alpha}[\nabla_{i\alpha}, B]_\pm + \overline{\xi}^{i\dot\alpha}[\overline{\nabla}_{i\dot\alpha}, B]_\pm\Big|, \\
\delta_1\delta_2 b &= -(\xi_1)^{j\beta}(\xi_2)^{i\alpha}[\nabla_{j\beta}, [\nabla_{i\alpha}, B]_\pm]_\mp - (\overline{\xi}_1)^{j\dot\beta}(\xi_2)^{i\alpha}[\overline{\nabla}_{j\dot\beta}, [\nabla_{i\alpha}, B]_\pm]_\mp \\
&\quad - (\xi_1)^{j\beta}(\overline{\xi}_2)^{i\dot\alpha}[\nabla_{j\beta}, [\overline{\nabla}_{i\dot\alpha}, B]_\pm]_\mp - (\overline{\xi}_1)^{j\dot\beta}(\overline{\xi}_2)^{i\dot\alpha}[\overline{\nabla}_{j\dot\beta}, [\overline{\nabla}_{i\dot\alpha}, B]_\pm]_\mp\Big| \qquad \text{(WZ gauge)},
\end{aligned}$$

so that

$$\begin{aligned}
(\delta_1\delta_2 - \delta_2\delta_1)b = &-i\Big((\xi_1)_i{}^\alpha(\sigma^\mu)_{\alpha\dot\alpha}(\overline{\xi}_2)^{i\dot\alpha} - (\xi_2)_i{}^\alpha(\sigma^\mu)_{\alpha\dot\alpha}(\overline{\xi}_1)^{i\dot\alpha}\Big)[\nabla_\mu, B] \\
&- i\Big((\xi_1)_{i\alpha}(\xi_2)^{i\alpha} + (\xi_1)_{i\dot\alpha}(\xi_2)^{i\dot\alpha}\Big)[\nabla_z, B] - i\Big((\xi_1)^i_\alpha(\xi_2)^{j\alpha} - (\xi_1)^i_{\dot\alpha}(\xi_2)^{j\dot\alpha}\Big)[W_{ij}, B]\Big|.
\end{aligned}$$

Thus, for the component fields $b = \phi_{ij},\ \lambda_{i\alpha},\ \overline{\lambda}_{i\dot\alpha},\ H_{ij},\ V_\mu$, we find

$$(\delta_1\delta_2 - \delta_2\delta_1)b = -i\Big((\xi_1)_i{}^\alpha(\sigma^\mu)_{\alpha\dot\alpha}(\overline{\xi}_2)^{i\dot\alpha} - (\xi_2)_i{}^\alpha(\sigma^\mu)_{\alpha\dot\alpha}(\overline{\xi}_1)^{i\dot\alpha}\Big)[\mathcal{D}_\mu, b]$$

$$- i\Big((\xi_1)_{i\alpha}(\xi_2)^{i\alpha} + (\xi_1)_{i\dot{\alpha}}(\xi_2)^{i\dot{\alpha}}\Big)[\partial_z, b] + i\Big[\Big((\xi_1)^i_\alpha(\xi_2)^{j\alpha} - (\xi_1)^i_{\dot{\alpha}}(\xi_2)^{j\dot{\alpha}}\Big)\phi_{ij}, b\Big], \tag{4.129}$$

where $\mathcal{D}_\mu$ is the ordinary gauge covariant derivative showing that the ∂_μ in eq. (4.128) is correctly gauge covariantized. The last term is the gauge transformation of the adjoint field b with the gauge parameter being $((\xi_1)^i_\alpha(\xi_2)^{j\alpha} - (\xi_1)^i_{\dot{\alpha}}(\xi_2)^{j\dot{\alpha}})\phi_{ij}$. Therefore the algebra closes up to a gauge transformation. This term appears due to the fact we have taken the WZ gauge. On the other hand, for $b = A_\mu$, we should take $B = i\nabla_\mu$ then we find

$$\begin{aligned}(\delta_1\delta_2 - \delta_2\delta_1)A_\mu = &-\Big[\mathcal{D}_\mu, \Big((\xi_1)^i_\alpha(\xi_2)^{j\alpha} - (\xi_1)^i_{\dot{\alpha}}(\xi_2)^{j\dot{\alpha}}\Big)\phi_{ij}\Big] \\ &- i\Big((\xi_1)_{i\alpha}(\xi_2)^{i\alpha} + (\xi_1)_{i\dot{\alpha}}(\xi_2)^{i\dot{\alpha}}\Big)[\partial_z, A_\mu] \\ &- i\Big((\xi_1)_i{}^\alpha(\sigma^\nu)_{\alpha\dot{\alpha}}(\overline{\xi}_2)^{i\dot{\alpha}} - (\xi_2)_i{}^\alpha(\sigma^\nu)_{\alpha\dot{\alpha}}(\overline{\xi}_1)^{i\dot{\alpha}}\Big)F_{\nu\mu},\end{aligned} \tag{4.130}$$

so that the last term is again a gauge transformation of the gauge field A_μ w.r.t. the gauge parameter $((\xi_1)^i_\alpha(\xi_2)^{j\alpha} - (\xi_1)^i_{\dot{\alpha}}(\xi_2)^{j\dot{\alpha}})\phi_{ij}$. Thus for any component field in this model the superalgebra closes off-shell up to a gauge transformation.

One may suspect that the above algebraic proof of the off-shell closure does not suffice to show the explicit supertransformations (4.116)–(4.121) certainly close. However, those transformation laws have been computed purely algebraically using only the constraints and the Bianchi identities, and the computation in the proof above has also been done in exactly the same manner. Thus the closeness is certainly and manifestly assured.

5 Conclusion and Discussion

We have seen how a superspace formulation using superconnections and supercurvatures would be set up for the $USp(4)$ super Yang-Mills theory in four dimensions. We introduced a central charge to prohibit unpleasant fields with spins higher than one appearing in the super Yang-Mills multiplet. In order to consider $N = 4$ supersymmetry with a central charge, we had to break the R-symmetry $SU(4)$ into some automorphic subgroup. We chose $USp(4)$ as such automorphic subgroup. We then obtained almost uniquely the appropriate constraints for supercurvatures by noticing the trivial algebraic symmetries, dropping the unnecessary supercurvatures which contain spins higher than one, and imposing the reality conditions on the supercurvatures to make the multiplet irreducible. These constraints have been consistently solved with the restrictions by the Bianchi identities. Then we have found that the theory we set up has one vector, four Weyl spinors, five real scalars, five auxiliary scalars and one extra vector-like field, which thus contains the same number of bosons and fermions, namely, 3+5+5+3=16 bosons and $4 \cdot 2 \cdot 2$=16 fermions, as the off-shell degrees of freedom. Supertransformations as well as transformations associated with the central charge can be computed using the Bianchi identities, and has been automatically shown to be off-shell in section 4. Thus concerning to supertransformations, we have succeeded to develop a superspace formulation for the $USp(4)$ super Yang-Mills theory in four dimensions.

There are, however, some problems remained. First, we have to notice the vector-like field V_μ noted above. By definition, this field has (mass) dimension two, the same as that of the auxiliary fields. This means that in four dimensions the field can not have a kinetic term as an elementary field, or, in other words, terms only like $V_\mu V^\mu/2$ can be allowed. One may then consider that this field should merely be an auxiliary or nondynamical field. We expect it is

not [12], in fact, the field is further constrained by eq. (4.101), i.e.

$$[\nabla^\mu, g_\mu] = -\{G_{i\alpha}, G^{i\alpha}\} - \{\overline{G}_{i\dot\alpha}, \overline{G}^{i\dot\alpha}\} + \frac{i}{4}[W^{ij}, [\nabla_z, W_{ij}]],$$

which is the same as eq. (4.25) in [12]. If we restrict the gauge group to be Abelian, this equation leads to

$$[\nabla^\mu, g_\mu] = 0. \tag{5.1}$$

Then as a solution we can take

$$g_\mu = [\nabla^\nu, (*A)_{\mu\nu}], \quad (*A)_{\mu\nu} + (*A)_{\nu\mu} = 0. \tag{5.2}$$

(Here the Hodge star is taken to make an gauge invariant quantity.) Thus the constraint (4.101) assures, in a way, that the field $V_\mu = ig_\mu|$ is not elementary and does contain contain dynamical degrees as in $V_\mu V^\mu = [\nabla^\nu, (*A)_{\mu\nu}][\nabla^\rho, (*A)^\mu{}_\rho\}]$ in the Abelian case. Similarly even in the non-Abelian cases, the field V_μ should be interpreted as non-elementary and dynamical degrees of freedom due to the constraint eq. (4.101). Here again we emphasize that eq. (4.101) is implemented in our formulation and derived automatically from the Bianchi identities.

Another crucial issue is whether we can construct the manifestly invariant action of the $USp(4)$ model within the framework of our superspace formulation. Since we have the component formulation (4.13), we should in principle construct the action on the superspace. In fact, as is mentioned in [12], it can be shown, by explicitly evaluating the components, that the action can be constructed in the form, roughly in our notation, like

$$S \sim \mathrm{tr} \int d^4x \, \left(\{\nabla, [\nabla, GG]\} + \{\overline{\nabla}, [\overline{\nabla}, \overline{GG}]\}\right)\big| .$$

However, because of the fact that $N = 4$ supersymmetry contains total of 16 supercharges, it may not possible to construct a manifestly supersymmetric action on superspace in four-dimensional spacetime. Moreover we work on the theory with a central charge, which makes its superspace formulation to develop much harder than usual. Nevertheless, we expect such superspace formulation could be established since the off-shell component formulation as its counterpart has been presented in a simple form. We have to reveal these points in the work in progress.

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A Notation and Formulae

A.1 Notation

We denote the symmetric group of degree p[19] by $\mathfrak{S}_p$. The signature for $\sigma \in \mathfrak{S}_p$ is defined as usual, i.e.

$$\mathrm{sgn}(\sigma) := \begin{cases} +1, & (\sigma : \text{even permutation}) \\ -1, & (\sigma : \text{odd permutation}) \end{cases} . \tag{A.1}$$

[19]It is the group consists of all bijective maps on the set $N_p := \{1, \cdots, p\}$.

We use the following notation for symmetrization and antisymmetrization of any kind of indices as

$$A^{(i_1\cdots i_p)} := \sum_{\sigma\in\mathfrak{S}_p} A^{i_{\sigma(1)}\cdots i_{\sigma(p)}}, \quad A^{[i_1\cdots i_p]} := \sum_{\sigma\in\mathfrak{S}_p} \mathrm{sgn}(\sigma) A^{i_{\sigma(1)}\cdots i_{\sigma(p)}}. \tag{A.2}$$

The set of all n by n matrices whose matrix components belong to the field K is denoted by $M(n,K)$. The $n\times n$ unit matrix on K is denoted as $\mathbf{1}_n \in M(n,K)$ or simply as $\mathbf{1}\in M(n,K)$. We denote the Pauli matrices by $\tau^i \in M(2,\mathbb{C})$ $(i=1,2,3)$. We have as usual

$$\tau^i\tau^j = \delta^{ij}\mathbf{1} + i\sum_{k=1}^{3}\epsilon^{ijk}\tau^k, \qquad \tau^2\tau^i\tau^2 = -(\tau^i)^*, \tag{A.3}$$

where ϵ^{ijk} is the totally antisymmetric symbol with $\epsilon^{123}=1$.

Through this article, we adopt the Einstein's convention, unless otherwise noted, to contract two identical indices with one from upper and the other from lower.

A.2 Weyl-Spinor Representations in Four Dimensions

A.2.1 Vector Basis in $M(2,\mathbb{C})$ — Minkowski Basis

Let

$$\sigma^\mu := (\mathbf{1}, \tau^i), \quad \bar\sigma^\mu := (\mathbf{1}, -\tau^i), \tag{A.4}$$

and

$$C_\downarrow := \tau^2, \quad C^\uparrow := ({C_\downarrow}^T)^{-1} \equiv -\tau^2, \quad {C_\downarrow}^* := (C_\downarrow)^* \equiv -\tau^2, \quad C^{\uparrow *} := (C^\uparrow)^* \equiv (({C_\downarrow}^*)^T)^{-1} \equiv \tau^2. \tag{A.5}$$

Note that the last four matrices are all antisymmetric so that satisfy

$$C^\uparrow = -(C_\downarrow)^{-1}, \quad C^{\uparrow *} = -({C_\downarrow}^*)^{-1}. \tag{A.6}$$

Two matrices σ^μ and $\bar\sigma^\mu$ satisfy the relations

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}, \quad \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = 2\eta^{\mu\nu}, \tag{A.7}$$

where $\eta^{\mu\nu} = (+,-,-,-)$ is the Minkowski metric in four dimensions, and are related by a transformation

$$(\bar\sigma^\mu)^T = C^\uparrow\sigma^\mu(C^{\uparrow *})^T, \quad \text{or} \quad (\sigma^\mu)^T = {C_\downarrow}^*\bar\sigma^\mu(C_\downarrow)^T. \tag{A.8}$$

Notice also that

$$(\sigma^\mu)^\dagger = \sigma^\mu, \quad (\bar\sigma^\mu)^\dagger = \bar\sigma^\mu. \tag{A.9}$$

We prescribe that, according to the reason which will be understood later, these matrices are labeled by row and column indices as

$$(\sigma^\mu)_{\alpha\dot\beta}, \quad (\bar\sigma^\mu)^{\dot\alpha\beta} \tag{A.10}$$

and

$$(C_\downarrow)_{\alpha\beta} =: C_{\alpha\beta}, \quad (C^\uparrow)^{\alpha\beta} =: C^{\alpha\beta}, \quad ({C_\downarrow}^*)_{\dot\alpha\dot\beta} := C_{\dot\alpha\dot\beta}, \quad (C^{\uparrow *})^{\dot\alpha\dot\beta} := C^{\dot\alpha\dot\beta}. \tag{A.11}$$

Then the above equations (A.6), (A.7), and (A.8) can be expressed by

$$C_{\alpha\beta}C^{\gamma\beta} = {\delta_\alpha}^\gamma, \quad C_{\dot\alpha\dot\beta}C^{\dot\gamma\dot\beta} = {\delta_{\dot\alpha}}^{\dot\beta}; \tag{A.12}$$

$$(\sigma^\mu)_{\alpha\dot\gamma}(\bar\sigma^\nu)^{\dot\gamma\beta} + (\sigma^\nu)_{\alpha\dot\gamma}(\bar\sigma^\mu)^{\dot\gamma\beta} = 2\eta^{\mu\nu}{\delta_\alpha}^\beta, \quad (\bar\sigma^\mu)^{\dot\alpha\gamma}(\sigma^\nu)_{\gamma\dot\beta} + (\bar\sigma^\nu)^{\dot\alpha\gamma}(\sigma^\mu)_{\gamma\dot\beta} = 2\eta^{\mu\nu}{\delta^{\dot\alpha}}_{\dot\beta}; \tag{A.13}$$

$$(\bar\sigma^\mu)^{\dot\alpha\beta} = C^{\beta\delta}C^{\dot\alpha\dot\gamma}(\sigma^\mu)_{\delta\dot\gamma}, \quad (\sigma^\mu)_{\alpha\dot\beta} = C_{\dot\beta\dot\delta}C_{\alpha\gamma}(\bar\sigma^\mu)^{\dot\delta\gamma}, \tag{A.14}$$

respectively.

Orthonormality and Completeness Using equations (A.7) and (A.8), we obtain the orthonormality of the matrices σ^μ, $\bar{\sigma}^\mu$:

$$\operatorname{tr}\sigma^\mu\bar{\sigma}^\nu = 2\eta^{\mu\nu}. \tag{A.15}$$

Then we can easily show that four matrices $\sigma^\mu \in M(2,\mathbb{C})$ are linearly independent, and because of the fact that $\dim M(2,\mathbb{C}) = 4$, they form a basis of the complex vector space $M(2,\mathbb{C})$. Similarly, four matrices $\bar{\sigma}^\mu \in M(2,\mathbb{C})$ form another basis of $M(2,\mathbb{C})$. Completeness of these basis can be expressed as

$$(\sigma^\mu)_{\alpha\dot{\gamma}}(\bar{\sigma}_\mu)^{\dot{\delta}\beta} = 2\delta_\alpha{}^\beta\delta^{\dot{\delta}}{}_{\dot{\gamma}}. \tag{A.16}$$

So called the Firtz transformations can be derived from the last identity; for instance,

$$(\sigma^\mu)_{\alpha\dot{\gamma}}(\sigma_\mu)_{\beta\dot{\delta}} = 2C_{\alpha\beta}C_{\dot{\gamma}\dot{\delta}}, \quad (\bar{\sigma}^\mu)^{\dot{\alpha}\gamma}(\bar{\sigma}_\mu)^{\dot{\beta}\delta} = 2C^{\dot{\alpha}\dot{\beta}}C^{\gamma\delta}. \tag{A.17}$$

Some Formulae From eq. (A.7) we obtain

$$\sigma^\mu\bar{\sigma}^\rho\sigma^\nu + \sigma^\nu\bar{\sigma}^\rho\sigma^\mu = 2(\eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\nu}\sigma^\rho), \tag{A.18}$$
$$\bar{\sigma}^\mu\sigma^\rho\bar{\sigma}^\nu + \bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\mu = 2(\eta^{\mu\rho}\bar{\sigma}^\nu + \eta^{\nu\rho}\bar{\sigma}^\mu - \eta^{\mu\nu}\bar{\sigma}^\rho). \tag{A.19}$$

Total antisymmetricity w.r.t. $\mu,\ \nu,\ \rho,\ \sigma$ in

$$\operatorname{tr}(\sigma^\mu\bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\sigma - \sigma^\nu\bar{\sigma}^\rho\sigma^\mu\bar{\sigma}^\sigma), \quad \operatorname{tr}(\bar{\sigma}^\mu\sigma^\rho\bar{\sigma}^\nu\sigma^\sigma - \bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\mu\sigma^\sigma) \tag{A.20}$$

leads to the equations

$$\sigma^\mu\bar{\sigma}^\rho\sigma^\nu - \sigma^\nu\bar{\sigma}^\rho\sigma^\mu = 2i\varepsilon^{\mu\nu\rho\sigma}\sigma_\sigma, \tag{A.21}$$
$$\bar{\sigma}^\mu\sigma^\rho\bar{\sigma}^\nu - \bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\mu = -2i\varepsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\sigma. \tag{A.22}$$

A.2.2 (Anti-) Symmetric Basis in $M(2,\mathbb{C})$ — Minkowski Basis

Let

$$\sigma^{\mu\nu} := \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu), \quad \bar{\sigma}^{\mu\nu} := \frac{1}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu). \tag{A.23}$$

Clearly, these matrices have standard index structure

$$(\sigma^{\mu\nu})_\alpha{}^\beta, \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{A.24}$$

and are traceless

$$\operatorname{tr}\sigma^{\mu\nu} = (\sigma^{\mu\nu})_\alpha{}^\alpha = 0, \quad \operatorname{tr}\bar{\sigma}^{\mu\nu} = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\alpha}} = 0. \tag{A.25}$$

Thus if we define matrices

$$(\sigma^{\mu\nu})_{\alpha\beta} := (\sigma^{\mu\nu})_\alpha{}^\gamma C_{\gamma\beta}, \quad (\sigma^{\mu\nu})^{\alpha\beta} := C^{\alpha\gamma}(\sigma^{\mu\nu})_\gamma{}^\beta, \tag{A.26}$$
$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} := C^{\dot{\beta}\dot{\gamma}}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\gamma}}, \quad (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} := (\bar{\sigma}^{\mu\nu})^{\dot{\gamma}}{}_{\dot{\beta}}C_{\dot{\gamma}\dot{\alpha}}, \tag{A.27}$$

they are all symmetric w.r.t. $\alpha,\ \beta$ or $\dot{\alpha},\ \dot{\beta}$. This fact can also be recognized by

$$\sigma^{\mu\nu}C_\downarrow = \frac{1}{2}\left(\sigma^\mu C^{\uparrow *}(\sigma^\nu)^T + (\sigma^\mu C^{\uparrow *}(\sigma^\nu)^T)^T\right), \quad \text{etc.} \tag{A.28}$$

Note also that

$$(\sigma^{\mu\nu})^\dagger = -\bar{\sigma}^{\mu\nu}, \quad (\bar{\sigma}^{\mu\nu})^\dagger = -\sigma^{\mu\nu}. \tag{A.29}$$

(Anti-) Self-duality Since $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ are traceless, each of them represents $4-1=3$ d.o.f. in $M(2,\mathbb{C})$, which corresponds to that $\sigma^{\mu\nu}C_{\downarrow}$, $\bar{\sigma}^{\mu\nu}(C^{\uparrow})^T$, etc., are symmetric and have $2\cdot 3/2=3$ d.o.f. These facts then imply that $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ are (anti-) self-dual "tensors". This is the case as we can show using the formulae above that

$$\tilde{\sigma}^{\mu\nu} := \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = i\sigma^{\mu\nu}, \quad \tilde{\bar{\sigma}}^{\mu\nu} := \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} = -i\bar{\sigma}^{\mu\nu}. \tag{A.30}$$

Orthonormality and Completeness Using eqs. (A.18), (A.21), and (A.15) we find that

$$\operatorname{tr}\sigma^{\mu\nu}\sigma^{\rho\sigma} = -4\wp^{+\mu\nu\rho\sigma}, \quad \operatorname{tr}\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma} = -4\wp^{-\mu\nu\rho\sigma}, \tag{A.31}$$

where

$$\wp^{+\mu\nu\rho\sigma} := \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho} - i\varepsilon^{\mu\nu\rho\sigma}), \quad \wp^{-\mu\nu\rho\sigma} := \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho} + i\varepsilon^{\mu\nu\rho\sigma}). \tag{A.32}$$

Two symbols $\wp^+$ and $\wp^-$ can be interpreted as the projection operators in four dimensions into the self-dual and anti-self-dual tensors, respectively, in the vector representation, in fact,

$$\frac{1}{2}\wp^{\pm\mu\nu\rho\sigma}\wp^{\pm}{}_{\rho\sigma}{}^{\tau\lambda} = \wp^{\pm\mu\nu\tau\lambda}, \quad \frac{1}{2}\wp^{\pm\mu\nu\rho\sigma}\wp^{\mp}{}_{\rho\sigma}{}^{\tau\lambda} = 0, \tag{A.33}$$

$$\wp^{\pm\rho\sigma\mu\nu} = \wp^{\pm\mu\nu\rho\sigma}, \quad \wp^{\pm\nu\mu\rho\sigma} = \wp^{\pm\mu\nu\sigma\rho} = -\wp^{\pm\mu\nu\rho\sigma}. \tag{A.34}$$

Then we find that the sets of four matrices

$$(C_{\alpha\beta},\ (\sigma^{\mu\nu})_{\alpha\beta}), \quad (C_{\dot{\alpha}\dot{\beta}},\ (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}), \tag{A.35}$$

or their variant with some indices raised, are separately taken to be bases in $M(2,\mathbb{C})$. Note here that

$$\operatorname{tr} C_{\downarrow}C^{\uparrow} = -2, \quad \operatorname{tr}\sigma^{\mu\nu}C_{\downarrow} = 0, \quad \text{etc.} \tag{A.36}$$

Completeness of these bases is expressed by identities as

$$C_{\alpha\beta}C^{\gamma\delta} + \frac{1}{2}(\sigma^{\mu\nu})_{\alpha\beta}(\sigma_{\mu\nu})^{\gamma\delta} = 2\delta_{\alpha}{}^{\gamma}\delta_{\beta}{}^{\delta}, \tag{A.37}$$

$$C_{\dot{\alpha}\dot{\beta}}C^{\dot{\gamma}\dot{\delta}} + \frac{1}{2}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}_{\mu\nu})^{\dot{\gamma}\dot{\delta}} = 2\delta_{\dot{\alpha}}{}^{\dot{\gamma}}\delta_{\dot{\beta}}{}^{\dot{\delta}}. \tag{A.38}$$

With the use of these identities, we can show, for instance, the following Firtz transformations

$$C_{\alpha\beta}C_{\gamma\delta} = C_{\alpha\gamma}C_{\beta\delta} - C_{\alpha\delta}C_{\beta\gamma}, \quad \frac{1}{2}(\sigma^{\mu\nu})_{\alpha\beta}(\sigma_{\mu\nu})_{\gamma\delta} = C_{\alpha\gamma}C_{\beta\delta} + C_{\alpha\delta}C_{\beta\gamma},\ \text{etc.} \tag{A.39}$$

A.2.3 Weyl Spinors in Four-Dimensional Minkowski Space

Weyl spinors are elements of representation spaces of $SL(2,\mathbb{C})$ corresponding to the two fundamental representations $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$. We choose to label a spinor which transforms under $(\mathbf{2},\mathbf{1})$ as ψ_α, and a the other as $\overline{\psi}_{\dot{\alpha}}$. Their contragradient representations are thus represented by ψ^α and $\overline{\psi}^{\dot{\alpha}}$. We assume these spinors are Grassmann odd. Matrices $C_{\downarrow}$ and $C_{\downarrow}{}^*$, or their contragradient $C^{\uparrow}$, $C^{\uparrow *}$, can be taken as the $SL(2,\mathbb{C})$ invariant metrics. We choose (σ^μ) as a basis of the representation space for $(\mathbf{2},\mathbf{1})\otimes(\mathbf{1},\mathbf{2})$, and $(\bar{\sigma}^\mu)$ as its contragradient. This is why we have labeled as $C_{\alpha\beta}$, $C_{\dot{\alpha}\dot{\beta}}$, $C^{\alpha\beta}$, $C^{\dot{\alpha}\dot{\beta}}$ and $(\sigma^\mu)_{\alpha\dot{\beta}}$, $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}$. Transformations are then represented as follows

$$\psi'_\alpha = \exp(+\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu})_\alpha{}^\beta\psi_\beta, \quad \overline{\psi}'_{\dot{\alpha}} = \exp(-\frac{1}{4}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\overline{\psi}_{\dot{\beta}}, \tag{A.40}$$

$$\psi'^\alpha = \exp(-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu})_\beta{}^\alpha\psi^\beta, \quad \overline{\psi}'^{\dot\alpha} = \exp(+\frac{1}{4}\omega_{\mu\nu}\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\overline{\psi}^{\dot\beta}. \tag{A.41}$$

Thus indices can be raised or lowered by C's by the same rule as before, namely,

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \psi^\beta C_{\beta\alpha}. \tag{A.42}$$

Notice here that

$$\psi_\alpha\chi^\alpha = -\psi^\alpha\chi_\alpha = +\chi_\alpha\psi^\alpha. \tag{A.43}$$

We also find that

$$\exp(-\frac{1}{4}\omega_{\mu\nu}\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\alpha} = (\exp(+\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu})_\alpha{}^\beta)^\dagger, \quad \text{i.e.} \quad \overline{\psi}_{\dot\alpha} = (\psi_\alpha)^\dagger, \quad \overline{\psi}^{\dot\alpha} = (\psi^\alpha)^\dagger, \tag{A.44}$$

i.e., indices α are dotted to $\dot\alpha$ under Hermitian conjugation and vice versa, which is consistent to that we have labeled $(C^*)_{\dot\alpha\dot\beta}$, $(\sigma^\mu)^\dagger{}_{\alpha\dot\beta} = (\sigma^\mu)_{\alpha\dot\beta}$, etc.

Through this article, Hermitian conjugation, as well as transposition, always changes an order of, even of Grassmann, quantities. Thus for example,

$$(\psi_\alpha\chi^\alpha)^\dagger = (\psi_\alpha C^{\alpha\beta}\chi_\beta)^\dagger = (\chi_\beta)^\dagger(C^{\dot\eta\dot\alpha})(\psi_\alpha)^\dagger = \overline{\chi}_{\dot\beta}\overline{\psi}^{\dot\beta}. \tag{A.45}$$

We define a fermionic derivative by

$$\left\{\frac{\partial}{\partial\theta^\alpha}, \theta^\beta\right\} = \delta_\alpha{}^\beta, \quad \left\{\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}, \overline{\theta}^{\dot\beta}\right\} = \delta_{\dot\alpha}{}^{\dot\beta}, \tag{A.46}$$

so that, by taking the Hermitian conjugation of this equation, we find that[20]

$$\left(\frac{\partial}{\partial\theta^\alpha}\right)^\dagger = \frac{\partial}{\partial(\theta^\alpha)^\dagger} = \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}. \tag{A.47}$$

We also find that

$$C^{\alpha\beta}\frac{\partial}{\partial\theta^\beta} = -\frac{\partial}{\partial\theta_\alpha}, \quad \text{etc.} \tag{A.48}$$

A.2.4 Euclidean Basis and Weyl Spinors in Four-Dimensional Euclidean Space

Let

$$\sigma^\mu := (\mathbf{1}, i\tau^i), \quad \bar\sigma^\mu := (\mathbf{1}, -i\tau^i). \tag{A.49}$$

Then we have

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}, \quad \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = 2\eta^{\mu\nu}, \tag{A.50}$$

where $\eta^{\mu\nu} = (+,+,+,+)$ is the four-dimensional Euclidean metric. Most of the definitions and formulas in the preceding sections also hold here by replacing the Minkowski metric by the Euclidean metric, and also $\varepsilon^{\mu\nu\rho\sigma}$ by $i\varepsilon^{\mu\mu\rho\sigma}$. There are, however, some exceptions; note

$$(\sigma^\mu)^\dagger = \bar\sigma^\mu, \quad (\bar\sigma^\mu)^\dagger = \sigma^\mu, \tag{A.51}$$

so that

$$(\sigma^{\mu\nu})^\dagger = -(\sigma^{\mu\nu}), \quad (\bar\sigma^{\mu\nu})^\dagger = -(\bar\sigma^{\mu\nu}). \tag{A.52}$$

These equations and eqs. (A.40) and (A.41) implies that indices α, $\dot\alpha$ are raised or lowered under Hermitian conjugations, namely,

$$(\psi_\alpha)^\dagger = \psi^\alpha, \quad (\overline{\psi}_{\dot\alpha})^\dagger = \overline{\psi}^{\dot\alpha}, \quad \text{etc.} \tag{A.53}$$

[20]Note in passing that, since $[\partial_\mu, x^\nu] = \delta_\mu{}^\nu$, we have $\partial_\mu{}^\dagger = -\partial_\mu$.

A.3 $SU(N)$

A.3.1 Special Unitary Group $SU(N)$

The special unitary group $SU(N)$ is the group composed of all unitary transformations on a vector space without the trivial phase transformation group $U(1)$. For convenience, it is represented by matrices w.r.t. an orthonormalized basis, namely

$$SU(N) = \{(M_i{}^j) \in GL(N,\mathbb{C}) | (M^\dagger)_i{}^k M_k{}^j = \delta_i{}^j,\ \det M = 1\}. \tag{A.54}$$

A.3.2 Special Unitary Algebra $\mathfrak{su}(N)$

The special unitary algebra $\mathfrak{su}(N)$ is composed of elements $X \in \mathfrak{gl}(N,\mathbb{C})$ which satisfies that $\exp(itX) \in SU(N)$, namely,

$$\mathfrak{su}(N) = \{(X_i{}^j) \in \mathfrak{gl}(N,\mathbb{C}) | (X^\dagger)_i{}^j = X_i{}^j,\ \mathrm{tr}\, X = 0\}. \tag{A.55}$$

Note here that $\mathfrak{su}(N)$ should be a real vector space. As a consequence we find that $\dim SU(N) = \dim \mathfrak{su}(N) = N^2 - 1$.

Letting

$$\langle X, Y\rangle := \mathrm{tr}(X^\dagger Y), \quad X,\ Y \in M(n,\mathbb{C}), \tag{A.56}$$

we introduce a positive-definite Hermitian form on $M(N,\mathbb{C})$. Then, by taking a suitable linear combination, we can obtain an orthonormalized basis (X^a) in $\mathfrak{su}(N)$ such as

$$\langle X^a, X^b\rangle = \frac{1}{2}\delta^{ab}. \tag{A.57}$$

In addition, if we define $X^0 := \mathbf{1}/\sqrt{2N} \in M(N,\mathbb{C})$, N^2 matrices X^a $(a = 0, \cdots, N^2-1)$ satisfy this relation, so that they can be interpreted as an orthonormalized basis in $M(N,\mathbb{C})$. The corresponding completeness relation is readily computed from this relation as

$$\sum_{a=0}^{N^2-1} (X^a)_i{}^j (X^a)_k{}^l = \frac{1}{2}\delta_i{}^l \delta_k{}^j, \tag{A.58}$$

$$\sum_{a=1}^{N^2-1} (X^a)_i{}^j (X^a)_k{}^l = \frac{1}{2N}(N\delta_i{}^l \delta_k{}^j - \delta_i{}^j \delta_k{}^l). \tag{A.59}$$

Let (X^a) be an orthonormalized basis in $M(N,\mathbb{C})$, and let

$$F^{abc} := 2\langle X^a, X^b X^c\rangle = \langle X^a, \{X^b, X^c\}\rangle + \langle X^a, [X^b, X^c]\rangle. \tag{A.60}$$

Note $F^{abc} = F^{bca} = F^{cab}$. Since

$$\langle X^a, \{X^b, X^c\}\rangle^* = \langle \{X^b, X^c\}, X^a\rangle = \langle X^a, \{X^b, X^c\}\rangle, \tag{A.61}$$

$$\langle X^a, [X^b, X^c]\rangle^* = \langle [X^b, X^c], X^a\rangle = -\langle X^a, [X^b, X^c]\rangle, \tag{A.62}$$

they are identified as

$$\langle X^a, \{X^b, X^c\}\rangle = d^{abc} := \Re F^{abc}, \quad -i\langle X^a, [X^b, X^c]\rangle = f^{abc} := \Im F^{abc}, \tag{A.63}$$

where d^{abc} is totally symmetric while f^{abc} is totally antisymmetric and in particular $d^{ab0} = \delta^{ab}/\sqrt{2n}$, $f^{ab0} = 0$, so that

$$\{X^a, X^b\} = \frac{1}{\sqrt{2N}}\delta^{ab} + \sum_{c=1}^{N^2-1} d^{abc} X^c, \tag{A.64}$$

$$[X^a, X^b] = i \sum_{c=1}^{N^2-1} f^{abc} X^c. \tag{A.65}$$

We now construct a specific orthonormalized basis in $\mathfrak{su}(N)$ explicitly. Since $X^\dagger = X$ ($X \in \mathfrak{su}(N)$), we find

$$(X_{\mathrm{R}})^T = X_{\mathrm{R}}, \quad (X_{\mathrm{I}})^T + X_{\mathrm{I}} = 0, \qquad \text{where} \quad X_{\mathrm{R}} := \Re X, \quad X_{\mathrm{I}} := \Im X. \tag{A.66}$$

Thus as a basis, we can take the following:

$$(X^m)_{kl} := \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{i=1}^{m} \delta^i{}_k \delta^i{}_l - m \delta^{m+1}{}_k \delta^{m+1}{}_l \right), \quad (m = 1, \cdots, N-1), \tag{A.67}$$

$$(X^{ij}_{\mathrm{s}})_{kl} := \frac{1}{2}(\delta^i{}_k \delta^j{}_l + \delta^j{}_k \delta^i{}_l), \quad (X^{ij}_{\mathrm{a}})_{kl} := \frac{1}{2} i (\delta^i{}_k \delta^j{}_l - \delta^j{}_k \delta^i{}_l), \quad (1 \leqslant i < j \leqslant N), \tag{A.68}$$

where we have used fixed indices which do not obey the general transformation laws. Note especially the Cartan subalgebra is generated by $\{X^m\}$, so that weights are given as

$$\nu^m = \left(0, \cdots, 0, -\frac{m-1}{\sqrt{2(m-1)m}}, \overbrace{\frac{1}{\sqrt{2m(m+1)}}}^{m}, \cdots, \frac{1}{\sqrt{2(N-1)N}} \right) \in \mathbb{R}^{N-1},$$

$$(m = 1, \cdots, N-1), \tag{A.69}$$

$$\nu^N = \left(0, \cdots, 0, -\frac{N-1}{\sqrt{2(N-1)N}} \right) \in \mathbb{R}^{N-1}, \tag{A.70}$$

which satisfy that

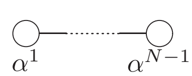
$$\|\nu^i\|^2 = \frac{N-1}{2N}, \quad \nu^i \cdot \nu^j = -\frac{1}{2N}, \quad (i \neq j), \quad \sum_{i=1}^{N} \nu^i = 0. \tag{A.71}$$

Correspondingly, simple roots are given as

$$\alpha^i := \nu^i - \nu^{i+1}, \quad (i = 1, \cdots, N-1), \tag{A.72}$$

$$\text{with} \quad \|\alpha^i\|^2 = 1, \quad \alpha^i \cdot \alpha^j = \begin{cases} 0, & (j \neq i, \ i \pm 1) \\ -\frac{1}{2}, & (j = i \pm 1) \end{cases}, \tag{A.73}$$

thus the Dynkin diagram for $\mathfrak{su}(N)$ is as follows:



(A.74)

A.3.3 Fundamental Representations

The fundamental representations (ρ, V) of $SU(N)$ are given by the N-dimensional defining representation eq. (A.54) and, more generally, its antisymmetric tensor products $(\rho \otimes \cdots \otimes \rho, \mathcal{A}(V \otimes \cdots \otimes V))$. This can be seen by the fact that the weight

$$\mu^j := \sum_{i=1}^{j} \nu^i \tag{A.75}$$

becomes fundamental. Their differential representations $(d\rho \oplus \cdots \oplus d\rho, \mathcal{A}(V \otimes \cdots \otimes V))$ become the fundamental representations of $\mathfrak{su}(N)$. We denote these representations by $[j] := \begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{j} \end{pmatrix}$. Since

$$\sum_{i=1}^{N} \nu^i = 0, \quad \text{i.e.} \quad \sum_{i=1}^{j} \nu^i = -\sum_{i=j+1}^{N} \nu^i, \tag{A.76}$$

we find that

$$\overline{[j]} = [N-j], \quad \text{especially} \quad \overline{\boldsymbol{N}} = [N-1]. \tag{A.77}$$

Adjoint representations $(\mathrm{Ad}, \mathfrak{su}(N))$ of $SU(N)$ and, as its differential representations, $(\mathrm{ad}, \mathfrak{su}(N))$ of $\mathfrak{su}(N)$, are N^2-1-dimensional and given from the fundamental representations as in

$$\boldsymbol{N} \otimes \overline{\boldsymbol{N}} = \boldsymbol{1} \oplus (\boldsymbol{N^2 - 1}). \tag{A.78}$$

A.4 $USp(2n)$

A.4.1 Symplectic Form

Let V be a complex vector space with dimension $\dim V = 2n$, and ω be a antisymmetric nondegenerate bilinear form[21] on V.

The existence of a nondegenerate bilinear form allows us to define a linear isomorphism $G : V \to V^*$ such that

$$^\forall v \in V, \quad \tilde{v} := G(v) \quad \text{s.t.} \quad ^\forall w \in V, \quad \tilde{v}(w) = \omega(v, w). \tag{A.79}$$

Similarly the inverse G^{-1} can be used to define an induced bilinear form $\tilde{\omega}$ on V^* such that

$$^\forall \phi \in V^*, \quad \tilde{\phi} := G^{-1}(\phi) \quad \text{s.t.} \quad ^\forall \psi \in V^*, \quad \psi(\tilde{\phi}) = \tilde{\omega}(\psi, \phi). \tag{A.80}$$

These operations are represented by components w.r.t. a basis (e^i) in V and its dual (e_i) in V^* as before; letting

$$\omega^{ij} := \omega(e^i, e^j), \quad \omega_{ij} := \tilde{\omega}(e_i, e_j), \quad v_i := e_i(v), \quad \phi^i := \phi(e^i), \quad v \in V, \ \phi \in V^*, \tag{A.81}$$

we find that

$$\omega_{ik}\omega^{jk} = \delta_i{}^j, \quad \tilde{v}^i = \omega^{ij} v_j, \quad \tilde{\phi}_i = \phi^j \omega_{ji}. \tag{A.82}$$

A.4.2 Unitary Symplectic Group $USp(2n)$

A transformation $f \in \mathrm{End}(V)$ such that

$$\omega(f(v), f(w)) = \omega(v, w), \quad \text{for}^\forall v, \ w \in V, \tag{A.83}$$

is called a symplectic transformation. The nondegeneracy of the symplectic form ω implies that $f \in \mathrm{Aut}(V)$. Then the set of all symplectic transformations on V forms a group, called the symplectic group on V. If V is complex and finite dimensional, as is the case in our assumption, one can impose that a symplectic transformation is simultaneously unitary w.r.t. a Hermitian

[21] A bilinear form $\omega : V \times V \to K$ is said to be antisymmetric if it satisfies that

$$^\forall v, \ w \in V, \quad \omega(v, w) + \omega(w, v) = 0,$$

i.e. $\omega \in \bigwedge^2 V^*$. An antisymmetric nondegenerate bilinear form ω exists only if the vector space V is even dimensional, since an antisymmetric matrix $(\omega(e^i, e^j)) \in M(\dim V, K)$, where (e^i) is a basis in V, cannot be nondegenerate (has at least one zero eigenvalue) when V is odd dimensional.

form on V. The collection of all both unitary and symplectic transformations on V also forms a group, which is called the unitary symplectic group, or the complex symplectic group, or simply, the symplectic group on the complex vector space V, and denoted as $USp(V)$ or simply, $Sp(V)$.

Let $f \in USp(V)$ and (e^i) be a basis in V. The condition (A.83) is represented w.r.t. the basis as in

$$M_k{}^i \omega^{kl} M_l{}^j = \omega^{ij}, \quad \text{equivalently}, \quad M_i{}^k \omega_{kl} M_j{}^l = \omega_{ij}, \tag{A.84}$$

where $f(e^i) = e^j M_j{}^i$. Notice here that such transformation satisfies $(\det M)^2 = 1$. Since V is even dimensional, we find that $\det M = 1$. Then assume the basis (e^i) is orthonormalized w.r.t. a Hermitian form

$$g(e^i, e^j) = \delta^{\bar{i}j}. \tag{A.85}$$

By a suitable orthogonal, but not symplectic in general, transformation of this basis, we can obtain a basis (e'^i) such that

$$\omega(e'^i, e'^j) = \Omega^{ij} := \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}. \tag{A.86}$$

We call this matrix representation of ω the standard form. The corresponding contragradient $\tilde{\omega}$ is represented

$$\tilde{\omega}(e'^i, e'^j) = \Omega_{ij} := \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \tag{A.87}$$

i.e. by the same matrix. Note that an orthogonal transformation is a unitary transformation, which in particular preserve the Hermitian form, so that (e'^i) is also orthonormalized. An orthonormalized basis which represents ω as the standard form is said to be standard.

The unitary symplectic group is usually represented w.r.t. a standard basis as in

$$USp(2n) = \{(M_i{}^j) \in GL(2n, \mathbb{C}) | M_k{}^i \Omega^{kl} M_l{}^j = \Omega^{ij},\ (M^\dagger)_i{}^k M_k{}^j = \delta_i{}^j\}. \tag{A.88}$$

If $M \in USp(2n)$, it automatically satisfies $\det M = 1$ as noted above.

In passing, we list some formulae in an irreducible matrix representation

$$\omega_\downarrow + (\omega_\downarrow)^T = 0, \quad \omega^\uparrow + (\omega^\uparrow)^T = 0, \quad \omega_\downarrow = ((\omega^\uparrow)^T)^{-1}, \quad (\omega_\downarrow)^\dagger = c(\omega_\downarrow)^{-1} = -c\omega^\uparrow, \tag{A.89}$$

where $0 \neq c \in \mathbb{R}$ and $\omega_\downarrow = (\omega_{ij})$ and $\omega^\uparrow = (\omega^{ij})$. In particular, the last equation is derived by comparing

$$M(\omega_\downarrow)^\dagger M^T = (\omega_\downarrow)^\dagger, \quad \text{and} \quad M(\omega_\downarrow)^{-1} M^T = (\omega_\downarrow)^{-1}, \tag{A.90}$$

and noting the uniqueness of such $\omega_\downarrow$ up to a constant as shown by the Schur's lemma for the irreducible representation. For the standard form, we find

$$(\Omega_\downarrow)^\dagger = (\Omega_\downarrow)^{-1} = -\Omega^\uparrow. \tag{A.91}$$

A.4.3 Unitary Symplectic Algebra $\mathfrak{usp}(2n)$

The unitary symplectic algebra $\mathfrak{usp}(2n)$ is the Lie algebra of the Lie group $USp(2n)$, namely, iff $X \in \mathfrak{usp}(2n)$ then $\exp(itX) \in USp(2n)$, so that

$$\mathfrak{usp}(2n) = \{(X_i{}^j) \in \mathfrak{gl}(2n, \mathbb{C}) | X_k{}^i \Omega^{kj} + \Omega^{ik} X_k{}^j = 0,\ (X^\dagger)_i{}^j = X_i{}^j\}. \tag{A.92}$$

Here again the condition $\operatorname{tr} X = 0$ $(X \in \mathfrak{usp}(2n))$ is automatically assured. If we write an element $X \in \mathfrak{usp}(2n)$ in the form

$$X = \begin{pmatrix} T & S \\ S' & T' \end{pmatrix}, \qquad T,\ T',\ S,\ S' \in \mathfrak{gl}(n, \mathbb{C}), \tag{A.93}$$

we find that

$$T^\dagger = T, \quad S^T = S, \quad T' = -T^T, \quad S' = S^\dagger. \tag{A.94}$$

Thus as a basis in $\mathfrak{usp}(2n)$, we can take the following

$$X^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad X^a = \begin{pmatrix} T^a & 0 \\ 0 & -(T^a)^* \end{pmatrix}, \quad T^a \in \mathfrak{su}(n), \quad (a = 1, \cdots, n^2 - 1), \tag{A.95}$$

$$X_{\mathrm{s}}^a = \begin{pmatrix} 0 & S_{\mathrm{R}} \\ S_{\mathrm{R}} & 0 \end{pmatrix}, \quad X_{\mathrm{a}}^a = \begin{pmatrix} 0 & -iS_{\mathrm{I}} \\ iS_{\mathrm{I}} & 0 \end{pmatrix}, \quad S_{\mathrm{R,I}} \in \mathfrak{gl}(n, \mathbb{R}), \quad S_{\mathrm{R,I}}^T = S_{\mathrm{R,I}},$$

$$(a = 1, \cdots, \frac{1}{2}n(n+1)), \tag{A.96}$$

and, as a consequence, we find that $\dim USp(2n) = \dim \mathfrak{usp}(2n) = n(2n+1)$.

We consider the Cartan subalgebra to be generated by the following n elements:

$$\tilde{X}^0 := \frac{1}{\sqrt{2n}} X^0, \quad \tilde{X}^m := \begin{pmatrix} X^m & 0 \\ 0 & -X^m \end{pmatrix}, \quad (m = 1, \cdots, n-1), \tag{A.97}$$

where X^m is given by (A.67). Then, using the weights (A.69), (A.70) and $\nu^{n+1} := e^{n+1} = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$, the corresponding weights are given

$$\pm\left(\nu^i + \frac{1}{\sqrt{2n}}\nu^{n+1}\right), \tag{A.98}$$

so that the roots are

$$\nu^i - \nu^j, \ (i \neq j), \quad \pm\left(\nu^i + \nu^j + \sqrt{\frac{2}{n}}\nu^{n+1}\right), \tag{A.99}$$

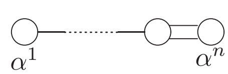
and thus simple roots are

$$\alpha^i = \begin{cases} \nu^i - \nu^{i+1}, & (i = 1, \cdots, n-1), \\ 2\nu^n + \sqrt{\frac{2}{n}}\nu^{n+1}, & (i = n), \end{cases} \tag{A.100}$$

which satisfy that

$$\|\alpha^i\|^2 = \begin{cases} 1, & (i = 1, \cdots, n-1), \\ 2, & (i = n), \end{cases} \quad \alpha^i \cdot \alpha^j = \begin{cases} -\frac{1}{2}, & (j = i+1, \ i = 1, \cdots, n-1, \text{ or } i \leftrightarrow j), \\ -1, & (i = n-1, \ j = n, \text{ or } i \leftrightarrow j), \\ 0, & (\text{otherwise}). \end{cases} \tag{A.101}$$

The Dynkin diagram for $\mathfrak{usp}(2n)$ is as follows:



(A.102)

This Dynkin diagram looks like that of $\mathfrak{so}(2n+1)$. However, norms of these simple roots are not the same with each other, thus the two diagrams are actually different, except the case for $n = 2$, so that $SO(5) \cong USp(4)$.

B Clifford Algebra and Spinors

In this appendix, we briefly examine some fundamental properties of Clifford algebra which plays a role in this article. Notation and convention follows the preceding appendix.

B.1 Clifford Algebra

B.1.1 Definition

Let η be a (non-degenerate) metric with (Sylvester's) signature (t, s) which is defined on a vector space $M(t, s)$ with dimension $D = t + s$ [47]:

$$\eta_{\mu\nu} = (\underbrace{+, \cdots, +}_{t}, \underbrace{-, \cdots, -}_{s}). \tag{B.1}$$

Clifford Algebra $\mathcal{C}(t, s)$ is defined to be an algebra which is generated by D elements $\{\gamma^1, \cdots, \gamma^D\}$ which satisfy the anticommutation relations[22]

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{B.2}$$

where $\eta^{\mu\nu}$ is defined to be the inverse of the metric $\eta_{\mu\nu}$:

$$\eta^{\mu\nu}\eta_{\nu\rho} = \eta^\mu{}_\rho \equiv \delta^\mu{}_\rho. \tag{B.3}$$

B.1.2 Irreducible Representation

In what follows, we will be only interested in a finite dimensional irreducible linear representation (ρ, V) of the Clifford algebra $\mathcal{C}(t, s)$ on a finite dimensional complex vector space V. We will therefore identify each element $\gamma \in \mathcal{C}(t, s)$ of the Clifford algebra with its image $\rho(\gamma) \in \mathrm{End}(V)$, which is a linear transformation on the vector space V, or a matrix on $\mathbb{C}^{\dim V}$, without distinguishing them. Further, we will only consider the case where the dimension D is even.

A basis of the algebra $\mathcal{C}(t, s)$ with even dimensions $D = t + s$ can be defined as

$$(\gamma^{\mu_1\cdots\mu_p})_{1\leqslant\mu_1<\cdots<\mu_p\leqslant D,\ 0\leqslant p\leqslant D}, \tag{B.4}$$

$$\gamma^{\mu_1\cdots\mu_p} := \begin{cases} 1 & (p = 0), \\ \dfrac{1}{p!}\displaystyle\sum_{\sigma\in\mathfrak{S}_p} \mathrm{sgn}(\sigma)\gamma^{\mu_{\sigma(1)}}\cdots\gamma^{\mu_{\sigma(p)}} & (p > 0), \end{cases} \tag{B.5}$$

where $\mathfrak{S}_p$ denotes the symmetric group of degree p. As a consequence, it follows that $\dim \mathcal{C}(t, s) = 2^D = 2^{D/2} \times 2^{D/2}$ and $\dim V = 2^{D/2}$.

That the elements in eq. (B.4) are linearly independent can be shown by using the orthogonality relations

$$\mathrm{tr}\, \gamma^{\mu_1\cdots\mu_p}\gamma^{\nu_1\cdots\nu_q} = (-1)^{p(p-1)/2} \dim V\, \delta_{pq}\eta_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_p} \tag{B.6}$$

where

$$\eta_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_p} := \sum_{\sigma\mathfrak{S}_p} \mathrm{sgn}(\sigma)\eta_{\mu_1\nu_{\sigma(1)}}\cdots\eta_{\mu_p\nu_{\sigma(p)}}, \tag{B.7}$$

which is essentially a direct consequence of the fact

$$\mathrm{tr}\, \gamma^{\mu_1\cdots\mu_p} = 0, \quad (1 \leqslant \mu_1 < \cdots < \mu_p \leqslant D,\ 1 \leqslant p \leqslant D). \tag{B.8}$$

[22] Let $T(M(t, s))$ be the tensor algebra on $M(t, s)$ and I_η be the two-sided ideal generated by the set

$$\{t \in T(M(t, s)) | t = v \otimes v - \eta(v, v),\ v \in M(t, s)\}.$$

Then Clifford algebra $\mathcal{C}(t, s)$ is defined as the quotient algebra $\mathcal{C}(t, s) = T(M(t, s))/I_\eta$. In $\mathcal{C}(t, s)$ tensor product of $\gamma,\ \gamma' \in \mathcal{C}(t, s)$ is simply denoted as $\gamma\gamma'$ instead of $\gamma \otimes \gamma'$. Clifford algebra is determined uniquely by the anticommutation relation $\{\gamma, \gamma'\} = 2\eta(\gamma, \gamma')$ up to an isomorphism due to the universality of tensor algebra. This fact is said to be the universality of Clifford algebra.

On the other hand, the completeness of the elements in eq. (B.4) can be proved by the fact that the Clifford algebra $\mathcal{C}(t,s)$ is generated by the elements $\{\gamma^1, \cdots, \gamma^D\}$. Note that by using the orthogonality eq. (B.6) the completeness relation can be given as the identity

$$\gamma = \sum_{p=0}^{D} \frac{1}{p!}(-1)^{p(p-1)/2}\frac{1}{\dim V}\gamma^{\mu_1\cdots\mu_p}\,\mathrm{tr}\,\gamma_{\mu_1\cdots\mu_p}\gamma, \quad \text{for}^{\forall}\gamma \in \mathcal{C}(t,s). \tag{B.9}$$

For later convenience, we define a special element (note here that $D = t + s$ is even and that so is $t - s$)

$$\Gamma^5 := i^{(t-s)/2}(-1)^s\gamma^1\cdots\gamma^D, \tag{B.10}$$

which satisfies that

$$\{\Gamma^5, \gamma^\mu\} = 0, \quad (\Gamma^5)^2 = 1. \tag{B.11}$$

We can also show that

$$\gamma^{\mu_1\cdots\mu_p}\Gamma^5 = \frac{1}{(D-p)!}i^{(t-s)/2}(-1)^{p(p-1)/2}\varepsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_D}\gamma^{\mu_{p+1}\cdots\mu_D}, \tag{B.12}$$

where the totally antisymmetric tensor $\varepsilon^{\mu_1\cdots\mu_D}$ is defined by

$$\varepsilon^{\mu_1\cdots\mu_D} = \frac{1}{|\det g|}\epsilon^{\mu_1\cdots\mu_D}, \qquad \epsilon^{1\cdots D} = 1. \tag{B.13}$$

B.1.3 Conjugations

We then consider the complex conjugation of the representation (ρ, V). Eq. (B.2) implies that

$$\{-(\gamma^\mu)^*, -(\gamma^\nu)^*\} = 2\eta^{\mu\nu}, \tag{B.14}$$

i.e. the complex conjugate $(-\rho^*, V)$ of the representation could also generate a representation of Clifford algebra on the same space V. The universality of the Clifford algebra[23] then requires the existence of a unitary transformation B s.t.[24]

$$B\gamma^\mu B^{-1} = \eta(\gamma^\mu)^*, \qquad B^\dagger B = 1, \qquad B^*B = \varepsilon, \qquad \eta = \pm 1, \quad \varepsilon = \pm 1. \tag{B.15}$$

Similarly, since

$$\{(-\gamma^\mu)^T, (-\gamma^\nu)^T\} = 2\eta^{\mu\nu}, \tag{B.16}$$

the contragradient representation $(-\rho^T, V)$ should be mapped by a unitary transformation C s.t.

$$C\gamma^\mu C^{-1} = \eta'(\gamma^\mu)^T, \qquad C^\dagger C = 1, \qquad C^T = \varepsilon' C, \qquad \eta' = \pm 1, \quad \varepsilon' = \pm 1. \tag{B.17}$$

We could also write

$$C^*B\gamma^\mu B^{-1}C^{*\,-1} = \eta(C\gamma^\mu C^{-1})^* = \eta\eta'(\gamma^\mu)^\dagger, \tag{B.18}$$

i.e.

$$\Gamma^0(\gamma^\mu)^\dagger(\Gamma^0)^{-1} = \kappa\gamma^\mu, \qquad (\Gamma^0)^\dagger\Gamma^0 = 1, \qquad \Gamma^0 := \varepsilon(-1)^{t(t-1)/2}B^{-1}C^{*\,-1}, \quad \kappa := \eta\eta', \tag{B.19}$$

[23] Clifford algebra is uniquely determined, up to an isomorphic algebra, by the fundamental anticommutation relations. This fact is called the universality of Clifford algebra, and is a consequence of the universality of tensor algebra.

[24] Since $\eta\gamma^\mu = B^{-1}(\gamma^\mu)^*B = B^*(\gamma^\mu)^*(B^{-1})^*$ the uniqueness of B requires $B^*B = \varepsilon$. Also it should be unitary, which can be similarly seen by the uniqueness of B and by the (anti-) Hermiticity of γ^μ shown in the following. All these analysis can be similarly applied to the case for C below.

(The sign $\varepsilon(-1)^{t(t-1)/2}$ which appears in the definition of Γ^0 is just for the later convenience.) which implies that $(\gamma^\mu)^\dagger = \kappa'_{(\mu)}\gamma^\mu$, $\kappa'_{(\mu)} = \pm 1$. This sign $\kappa'_{(\mu)}$ can not be chosen arbitrarily; for,

$$\begin{cases} (\gamma^{\rm t})^2 = +\mathbf{1}, \\ (\gamma^{\rm s})^2 = -\mathbf{1}, \end{cases} \quad \text{i.e.} \quad \begin{cases} (\gamma^{\rm t})^\dagger(\gamma^{\rm t}) = +\kappa'_{\rm t}\mathbf{1}, \\ (\gamma^{\rm s})^\dagger(\gamma^{\rm s}) = -\kappa'_{\rm s}\mathbf{1}, \end{cases} \tag{B.20}$$

and, for any matrix M, $M^\dagger M$ is positive semidefinite, so $\kappa'_{\rm t} = +1$ and $\kappa'_{\rm s} = -1$, i.e.,

$$\begin{cases} (\gamma^{\rm t})^\dagger = +(\gamma^{\rm t}), \\ (\gamma^{\rm s})^\dagger = -(\gamma^{\rm s}). \end{cases} \tag{B.21}$$

This leads to

$$\begin{cases} \Gamma^0 = \gamma^1 \cdots \gamma^t, \\ \kappa = (-1)^{t+1}, \end{cases} \quad \text{or} \quad \begin{cases} \Gamma^0 = \gamma^{t+1} \cdots \gamma^D, \\ \kappa = (-1)^s, \end{cases} \tag{B.22}$$

and also

$$\begin{cases} \eta' = (-1)^{t+1}\eta, \\ \varepsilon' = \varepsilon\eta^t(-1)^{t(t-1)/2}, \end{cases} \quad \text{or} \quad \begin{cases} \eta' = (-1)^s\eta, \\ \varepsilon' = \varepsilon\eta^s(-1)^{s(s-1)/2}. \end{cases} \tag{B.23}$$

These two choices are equivalent, so that, in what follows, we choose the former representation for Γ^0. Further, ε and η can be related as follows. Notice that

$$(C\gamma^{\mu_1\cdots\mu_p})^T = (-1)^{p(t+1)+p(p-1)/2+t(t-1)/2}\eta^{t+p}\varepsilon(C\gamma^{\mu_1\cdots\mu_p}), \tag{B.24}$$

hence $C\gamma^{\mu_1\cdots\mu_p}$ are either symmetric or antisymmetric. The total number of antisymmetric elements is

$$\#(\mathrm{AS}) = \frac{1}{2}2^{D/2}(2^{D/2}-1), \tag{B.25}$$

which can be also calculated as

$$\#(\mathrm{AS}) = \sum_p \begin{pmatrix} D \\ p \end{pmatrix} \tag{B.26}$$

where the summation is over the numbers p such that

$$(-1)^{p(t+1)+p(p-1)/2+t(t-1)/2}\eta^{t+p}\varepsilon = -1, \quad \text{i.e.} \quad (-1)^{p(t+1)+p(p-1)/2}\eta^p = -(-1)^{t(t-1)/2}\eta^t\varepsilon. \tag{B.27}$$

Equating eqs. (B.25) and (B.26), we obtain Letting $f(p) = (-1)^{p(p-1)/2+p(t+1)}\eta^p$, we note

$p \pmod 4$	$f(p)$
0	1
1	$(-1)^{t+1}\eta$
2	-1
3	$-(-1)^{t+1}\eta$

$(-1)^{t+1}\eta$	$f(p)$
1	$1, 1, -1, -1$
-1	$1, -1, -1, 1$

and, labeling the cases by $((-1)^{t+1}\eta, -(-1)^{t(t-1)}\eta^t\varepsilon)$, we can compute $\#(\mathrm{AS})$ as follows

(i) $(+,+)$

$$\begin{aligned} \#(\mathrm{AS}) &= \sum_{p=0}^{D} \begin{pmatrix} D \\ p \end{pmatrix} \frac{1}{2}\left(1 + \Re\sqrt{2}e^{-i\frac{\pi}{4}+i\frac{n\pi}{2}}\right) \\ &= \frac{1}{2}2^{D/2}\left(2^{D/2} - \sqrt{2}\cos\frac{\pi}{4}(D+3)\right). \end{aligned}$$

(ii) $(+,-)$

$$\#(\mathrm{AS}) = \sum_{p=0}^{D} \left(\begin{array}{c} D \\ p \end{array} \right) \frac{1}{2} \left(1 - \Re\sqrt{2} e^{-i\frac{\pi}{4} + i\frac{n\pi}{2}} \right)$$
$$= \frac{1}{2} 2^{D/2} \left(2^{D/2} - (-1)\sqrt{2} \cos \frac{\pi}{4}(D+3) \right).$$

(iii) $(-,+)$

$$\#(\mathrm{AS}) = \sum_{p=0}^{D} \left(\begin{array}{c} D \\ p \end{array} \right) \frac{1}{2} \left(1 + \Re\sqrt{2} e^{i\frac{\pi}{4} + i\frac{n\pi}{2}} \right)$$
$$= \frac{1}{2} 2^{D/2} \left(2^{D/2} - \sqrt{2} \cos \frac{\pi}{4}(-D+3) \right).$$

(iv) $(-,-)$

$$\#(\mathrm{AS}) = \sum_{p=0}^{D} \left(\begin{array}{c} D \\ p \end{array} \right) \frac{1}{2} \left(1 - \Re\sqrt{2} e^{i\frac{\pi}{4} + i\frac{n\pi}{2}} \right)$$
$$= \frac{1}{2} 2^{D/2} \left(2^{D/2} - (-1)\sqrt{2} \cos \frac{\pi}{4}(-D+3) \right).$$

Thus we find that

$$\#(\mathrm{AS}) = \frac{1}{2} 2^{D/2} \left(2^{D/2} - \left(-(-1)^{t(t-1)/2} \eta^t \varepsilon \right) \sqrt{2} \cos \frac{\pi}{4} \left(\eta(-1)^{t+1} D + 3 \right) \right). \tag{B.28}$$

Equating this to $\frac{1}{2} 2^{D/2}(2^{D/2}-1)$, we obtain

$$\varepsilon = -\sqrt{2}\eta^t (-1)^{t(t-1)/2} \cos \frac{1}{4}\pi \left(\eta(-1)^{t+1} D + 3 \right) = \cos \frac{1}{4}\pi(s-t) - \eta \sin \frac{1}{4}\pi(s-t). \tag{B.29}$$

Conjugations of Γ^5 are computed as

$$B\Gamma^5 B^{-1} = (-1)^{(t-s)/2} (\Gamma^5)^*, \quad C\Gamma^5 C^{-1} = (-1)^{D/2} (\Gamma^5)^T, \tag{B.30}$$
$$\Gamma^0 (\Gamma^5)^\dagger (\Gamma^0)^{-1} = (-1)^t \Gamma^5, \quad (\Gamma^5)^\dagger = \Gamma^5. \tag{B.31}$$

Note also that

$$B\Gamma^0 B^{-1} = \eta^t (\Gamma^0)^*, \quad C\Gamma^0 C^{-1} = \eta^t (-1)^{t(t-1)/2} (\Gamma^0)^T, \quad (\Gamma^0)^2 = (-1)^{t(t-1)/2} \mathbf{1}. \tag{B.32}$$

B.1.4 The $SO(t,s)$ Subalgebra, $Spin(t,s)$

Let

$$\Sigma^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu] \equiv \frac{i}{2}\gamma^{\mu\nu}. \tag{B.33}$$

Then it can be shown that $\{\Sigma^{\mu\nu}\}_{1\leqslant\mu<\nu\leqslant D}$ obeys the $\mathfrak{so}(t,s)$ commutation relations

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = i(\eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\nu\sigma}\Sigma^{\mu\rho} - \eta^{\mu\rho}\Sigma^{\nu\sigma} + \eta^{\mu\sigma}\Sigma^{\nu\rho}), \tag{B.34}$$

and thus generates the $\mathfrak{so}(t,s)$ subalgebra in $\mathcal{C}(t,s)$. In fact, this subalgebra generates a group,

$$Spin(t,s) := \left\{ \gamma \in \mathcal{C}(t,s) | \gamma = \exp\left(-i\frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu} \right) \right\}, \tag{B.35}$$

with

$$\gamma\gamma^\mu\gamma^{-1} = \exp(\omega)^\mu{}_\nu\gamma^\nu. \tag{B.36}$$

To show that $Spin(t,s)$ is actually the double cover of $SO(t,s)$, let us consider the adjoint representation on the vector space

$$W = \langle\{\gamma^1,\ \cdots,\ \gamma^D\}\rangle, \tag{B.37}$$

as in

$$\Gamma \in Spin(t,s),\ \gamma \in W, \qquad \mathrm{Ad}(\Gamma)(\gamma) = \Gamma\gamma\Gamma^{-1}. \tag{B.38}$$

Let $\mathrm{Ad}(\Gamma) = 1$. Then decompose $\Gamma = \Gamma_0 + \gamma^p\Gamma'$, where γ_0 and γ' do not contain Γ^p. Note here that Γ contains terms with only even number of γ^p's by definition. We thus find

$$\gamma^p = \Gamma\gamma^p\Gamma^{-1}, \quad \text{i.e.} \quad \gamma^p(\Gamma_0 + \gamma^p\Gamma') = (\Gamma_0 + \gamma^p\Gamma')\gamma^p, \tag{B.39}$$

so that

$$(\gamma^p)^2\Gamma' = \gamma^p\Gamma'\gamma^p = -(\gamma^p)^2\Gamma', \quad \text{i.s.} \quad \Gamma' = 0. \tag{B.40}$$

Thus Γ does not contain either of γ^p, so that $\Gamma \in \mathbb{C}$, hence $\Gamma = \pm 1$.

Since (for even D)

$$[\Gamma^5, \Sigma^{\mu\nu}] = 0, \tag{B.41}$$

the representation of $Spin(t,s)$ on V is reducible, although the representation of $\mathcal{C}(t,s)$ as a whole is irreducible as noted before. This reducible representation can be decomposed into the two irreducible representations on the two eigenspaces of Γ^5 with eigenvalues, ± 1, called the chirality of the representations. Projection operators onto these two eigenspace are given as

$$P^\pm = \frac{1 \pm \Gamma^5}{2}, \tag{B.42}$$

where

$$P^\pm P^\pm = P^\pm, \quad P^\pm P^\mp = 0, \quad (P^\pm)^\dagger = P^\pm, \quad P^+ + P^- = 1. \tag{B.43}$$

Note that

$$[\Sigma^{2a-1,2a}, \Sigma^{2b-1,2b}] = 0. \tag{B.44}$$

Hence $\Sigma^{2a-1,2a}$ can form a Cartan subalgebra and can be simultaneously diagonalized. Let

$$S^a := (-i)^{\delta_{a,(t+1)/2} + \theta(a-(t+2)/2)}\Sigma^{2a-1,2a} = \Gamma^{a+}\Gamma^{a-} - \frac{1}{2}. \tag{B.45}$$

Then we find that S^a takes half-integer values and can be identified as the spin operators.

B.2 Spinors

B.2.1 $Spin(t,s)$ Spinors

We have seen that the Clifford algebra $\mathcal{C}(t,s)$ is irreducibly represented on $V = \mathbb{C}^{2^{D/2}}$. This spinor representation also represents $Spin(t,s)$, the double cover of $SO(t,s)$, which is not irreducible, and decomposes into two irreducible representations on the eigenspaces with chirality ± 1. We call these two irreducible representations of $Spin(t,s)$ the Weyl representations, and elements of the corresponding representation spaces Weyl spinors. Weyl spinors are thus given from a general $2^{D/2}$-component spinor $\psi \in V$ as

$$\psi^\pm = P^\pm\psi, \qquad P^\pm\psi^\pm = \pm\psi^\pm, \tag{B.46}$$

and can be considered as $2^{D/2-1}$-component objects. On the other hand, a $2^{D/2}$-component spinor $\psi \in V$, sometimes called a Dirac spinor, can be constructed from two Weyl spinors

$$\psi = (P^+ + P^-)\psi = \psi^+ + \psi^-, \tag{B.47}$$

$$\mathbf{2^{D/2}} = \mathbf{2^{D/2-1}} \oplus \mathbf{2^{D/2-1}}. \tag{B.48}$$

We choose to label a $2^{D/2}$-component spinor as ψ_α, so that $\gamma \in \mathcal{C}(t,s)$, including Γ^0 and Γ^5, is labeled as

$$\gamma_\alpha{}^\beta, \quad (\gamma^{-1})_\alpha{}^\beta, \quad (\gamma^\dagger)_\alpha{}^\beta, \quad (\gamma^T)^\alpha{}_\beta = \gamma_\beta{}^\alpha, \quad (\gamma^*)^\alpha{}_\beta = (\gamma_\alpha{}^\beta)^*, \tag{B.49}$$

and conjugation matrices $\mathfrak{C} = C,\ B$ are labeled as

$$\mathfrak{C}^{\alpha\beta}, \quad (\mathfrak{C}^{-1})_{\alpha\beta}, \quad (\mathfrak{C}^T)^{\alpha\beta} = \mathfrak{C}^{\beta\alpha}, \quad (\mathfrak{C}^*)_{\alpha\beta} = (\mathfrak{C}^{\alpha\beta})^*, \quad (\mathfrak{C}^\dagger)_{\alpha\beta} = (\mathfrak{C}^{\beta\alpha})^*. \tag{B.50}$$

Note that the indices are consistently set in

$$(\Gamma^0)_\alpha{}^\beta = (B^{-1})_{\alpha\gamma}(C^{*-1})^{\gamma\beta}. \tag{B.51}$$

Then a spinor transforms under $Spin(t,s)$ as

$$\psi'_\alpha = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)_\alpha{}^\beta\psi_\beta, \quad \text{or} \quad \delta\psi_\alpha = \frac{1}{4}\omega_{\mu\nu}(\gamma^{\mu\nu})_\alpha{}^\beta\psi_\beta. \tag{B.52}$$

Upper indices are thus label the contragradient representations, as in

$$\psi'^\alpha = \exp\left(+\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)_\beta{}^\alpha\psi^\beta, \quad \text{or} \quad \delta\psi^\alpha = -\frac{1}{4}\omega_{\mu\nu}(\gamma^{\mu\nu})_\beta{}^\alpha\psi^\beta. \tag{B.53}$$

B.2.2 Conjugations

Conjugations discussed above are used to define some conjugates of spinors.

First, we define the Dirac conjugate of $\psi \in V$ as

$$\overline{\psi}^\alpha := \sum_\beta (\psi_\beta)^\dagger((\Gamma^0)^{-1})_\beta{}^\alpha. \tag{B.54}$$

Note here that

$$\Gamma^0(\Sigma^{\mu\nu})^\dagger(\Gamma^0)^{-1} = \Sigma^{\mu\nu}. \tag{B.55}$$

The conjugate spinor[25] does transform contragradiently as indicated by the upper indices:

$$\overline{\psi}'^\alpha = (\psi_\gamma)^\dagger \exp\left(+\frac{i}{2}\omega^{\mu\nu}(\Sigma_{\mu\nu})^\dagger\right)_\gamma{}^\beta(\Gamma^0)^{-1\alpha}_\beta = \overline{\psi}^\beta \exp\left(+\frac{i}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}\right)_\beta{}^\alpha. \tag{B.56}$$

For an upper-index spinor ψ^α, the Dirac conjugate can be defined as

$$\overline{\psi}_\alpha := (\Gamma^0)^{-1}{}_\alpha{}^\beta(\psi^\beta)^\dagger, \tag{B.57}$$

so that successive Dirac conjugations reduces to the identity operation

$$\overline{\overline{\psi}} = (\Gamma^0)^{-1}(\overline{\psi})^\dagger = (\Gamma^0)^{-1}\Gamma^0\psi = \psi. \tag{B.58}$$

[25] In what follows we will omit the summation symbol for the Dirac conjugation.

The next one is the charge conjugation. We define it for a lower-index spinor,

$$\tilde{\psi}^{\alpha} := (\psi^T)_{\beta}(C^T)^{\beta\alpha} = C^{\alpha\beta}\psi_{\beta}, \tag{B.59}$$

and for an upper-index spinor,

$$\tilde{\psi}_{\alpha} := (C^{-1})_{\alpha\beta}(\psi^T)^{\beta}. \tag{B.60}$$

Again we have defined these conjugations so that $\tilde{\tilde{\psi}} = \psi$. Since

$$C\Sigma^{\mu\nu}C^{-1} = -(\Sigma^{\mu\nu})^T, \tag{B.61}$$

these conjugates transform correctly as indicated by indices; for instance,

$$\tilde{\psi}'_{\alpha} = (C^{-1})_{\alpha\beta} \exp\left(+\frac{i}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}\right)^{T}{}^{\beta}{}_{\gamma}(\psi^T)^{\gamma} = \exp\left(-\frac{i}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}\right){}_{\alpha}{}^{\beta}\tilde{\psi}_{\beta}. \tag{B.62}$$

Conventionally, the term charge conjugation may only be used for the operation to define

$$(\psi_c)_{\alpha} := (C^{-1})_{\alpha\beta}(\overline{\psi}^T)^{\beta} = \tilde{\overline{\psi}}_{\alpha}, \qquad \text{with} \qquad \psi_{cc} = \varepsilon\psi. \tag{B.63}$$

Here we also define the charge conjugate with upper index

$$(\psi_c)^{\alpha} := C^{\alpha\beta}\overline{\psi}_{\beta} = \tilde{\overline{\psi}}^{\alpha}, \qquad \text{with} \qquad \psi_{cc} = \varepsilon\psi. \tag{B.64}$$

The last one is called the B-conjugation, defined with the matrix B as[26]

$$(\psi_b)_{\alpha} := (B^{-1})_{\alpha\beta}(\psi^*)^{\beta}, \qquad (\psi^*)^{\alpha} := (\psi_{\alpha})^*, \tag{B.65}$$

$$(\psi_b)^{\alpha} := \eta^t B^{\alpha\beta}(\psi^*)_{\beta}, \qquad (\psi^*)_{\alpha} := (\psi^{\alpha})^*, \tag{B.66}$$

with $\psi_{bb} = \varepsilon\psi$. Notice that the B-conjugation is actually the same operation as the charge conjugation[27], i.e., $\psi_b = \psi_c$.

B.2.3 Majorana Spinors

Generally a $2^{D/2}$-component Dirac spinor is, though irreducible w.r.t. the whole representation of the Clifford algebra, reducible w.r.t. the representation of the $Spin(t,s)$ subalgebra. Thus, as a representation of $Spin(t,s)$, it is appropriate to use the Weyl representations with $2^{D/2-1}$-component Weyl spinors, which is irreducible w.r.t. $Spin(t,s)$. In such cases, it is sometimes more convenient to treats a pair of a Weyl spinor and its partner with opposite chirality as one $2^{D/2}$-component spinor, which is called a Majorana spinor. Note that d.o.f. of the two Weyl spinors, one of which is a partner to the other, is $2 \times 2^{D/2-1} = 2^{D/2}$, while that of a general complex $2^{D/2}$-component spinor is $2 \times 2^{D/2}$. Thus we need to impose a reality condition on a Majorana spinor to reduce the d.o.f. Such conditions are called the Majorana conditions. Of course, Majorana conditions have to be $Spin(t,s)$-covariant.

Let $(\psi_{i\alpha})$ $(i = 1, \cdots, N)$ be a multiplet of N Majorana spinors with an internal symmetry G labeled by the indices i. We adopt the following Majorana condition on such multiplets:

$$\psi_{i\alpha} = M_{ij}(\psi_c)^j{}_{\alpha}, \qquad M_{ij}M^{ik} = \delta_j{}^k, \tag{B.67}$$

[26]Remark here we have used somewhat misleading labeling for ψ^*'s; $(\psi^*)^{\alpha} = (\psi_{\alpha})^*$ does not necessarily transform as an upper-index spinor.

[27]This is due to the fact that there are essentially two kinds of conjugations in a complex vector space, one is the ordinary contragradient, and the other is the complex conjugation.

where

$$(\psi_c)^i{}_\alpha = (C^{-1})_{\alpha\beta}(\overline{\psi}^T)^{i\beta} = (C^{-1})_{\alpha\beta}((\Gamma^0)^{-1\ T})^\beta{}_\gamma(\psi^*)^{j\gamma}, \quad (\psi^*)^{i\alpha} := (\psi_{i\alpha}), \tag{B.68}$$

and M_{ij} is a G (-invariant) metric. Notice here that the charge conjugation matrix C is a $Spin(t,s)$-invariant metric, and the condition (B.67) is $Spin(t,s)$ and G-covariant. Since the condition (B.67) implies that

$$\psi_{i\alpha} = \varepsilon M_{ik}(M^*)^{kj}\psi_{j\alpha}, \tag{B.69}$$

so that

$$M_{ik}(M^*)^{kj} = \varepsilon, \qquad \text{i.e.,} \qquad M^\dagger = \varepsilon(M^{-1})^T. \tag{B.70}$$

Thus the Majorana condition (B.67) can be possibly imposed only if there exists a metric M_{ij} which satisfy the condition (B.70). Since such a metric does not necessarily exist for any internal symmetry group the Majorana condition restricts the possible internal symmetries. This fact is seen more specifically as follows. For simplicity, we assume a unitary representation for the internal symmetry metric, namely, $M^\dagger = M^{-1}$. Then the condition (B.70) is written as $M = \varepsilon M^T$.

(i) If $\varepsilon = +1$, the metric has to be symmetric: $M = M^T$. If further M is supposed to be invariant w.r.t. the internal symmetry, we may take G to be (a subgroup of) $O(N)$. Majorana spinors which obey the Majorana condition with $\varepsilon = +1$ and $G \cong O(N)$ are thus called $O(N)$-Majorana spinors.
Especially, if $N = 1$, i.e., if we consider singlet Majorana spinors with no internal symmetry, $M = 1$ so that only the Majorana condition with $\varepsilon = +1$ is possible. In other words, if $\varepsilon = -1$, we can not consider singlet Majorana spinor.

(ii) If $\varepsilon = -1$, the metric has to be antisymmetric: $M + M^T = 0$. If further M is supposed to be invariant w.r.t. the internal symmetry, we may take G to be (a subgroup of)[28] $USp(2n)$ with $N = 2n$. Majorana spinors which obey the Majorana condition with $\varepsilon = -1$ and $G \cong USp(2n)$ are thus called $USp(2n)$-Majorana spinors. For instance, if $N = 2$, we can consider as the internal symmetry $USp(2) \cong SU(2)$ and $SU(2)$-Majorana condition with the invariant metric $C_{ij} = (\tau^2)_{ij}$.

In some cases, Weyl spinors could independently become Majorana (including G-Majorana) spinors, and, if so, are called Majorana-Weyl spinors[29]. The condition is that

$$(\psi^\pm)_{\rm c} = \psi^\pm, \tag{B.71}$$

which is possible only if

$$P^{\pm(-1)^t} = CP^\pm C^{-1} = P^{\pm(-1)^{D/2}}, \qquad \text{i.e.} \qquad P^{\pm\sigma} = \mathbb{1}, \qquad \sigma := (-1)^{(s-t)/2}, \tag{B.72}$$

so that $\sigma = 1$, or, $s - t \equiv 0 \pmod 4$.

C Solutions of the $USp(4)$ SYM

In this section, we follow the whole computations to solve the system of constraints (4.78)–(4.82), i.e.

$$\{\nabla_{i\alpha}, \nabla_{j\beta}\} = i\Omega_{ij}C_{\alpha\beta}\nabla_z - iC_{\alpha\beta}W_{ij}, \qquad \{\overline{\nabla}_{i\dot\alpha}, \overline{\nabla}_{j\dot\beta}\} = i\Omega_{ij}C_{\dot\alpha\dot\beta}\nabla_z + iC_{\dot\alpha\dot\beta}W_{ij},$$

[28] As a nontrivial candidate, we can take, for example, $G \cong Spin(4)$.

[29] Thus, each Weyl component (or Weyl decomposition) of a Majorana-Weyl spinor is itself real, or self-conjugate, while Weyl components of a general Majorana spinor are conjugate to each other. In this sense, the Weyl decomposition of a Majorana-Weyl spinor should be merely denoted as in $\mathbf{2}^{D/2} \to \mathbf{2}^{D/2-1} + \mathbf{2}'^{D/2-1}$, whereas that of a non-Weyl-Majorana spinor can be as in $\mathbf{2}^{D/2} \to \mathbf{2}^{D/2-1} + \bar{\mathbf{2}}^{D/2-1}$.

$$
\begin{aligned}
\{\nabla_{i\alpha}, \overline{\nabla}_{j\dot\beta}\} &= i\Omega_{ij}(\sigma^\mu)_{\alpha\dot\beta}\nabla_\mu, \\
[\nabla_{i\alpha}, \nabla_\mu] &= -iF_{i\alpha\mu}, & [\overline{\nabla}_{i\dot\alpha}, \nabla_\mu] &= +i\overline{F}_{i\dot\alpha\mu}, \\
[\nabla_{i\alpha}, \nabla_z] &= -iG_{i\alpha}, & [\overline{\nabla}_{i\dot\alpha}, \nabla_z] &= +i\overline{G}_{i\dot\alpha}, \\
[\nabla_\mu, \nabla_z] &= -ig_\mu, & [\nabla_\mu, \nabla_\nu] &= -iF_{\mu\nu}, \\
\Omega^{ij}W_{ij} &= 0, & (W_{ij})^* &= W^{ij},
\end{aligned}
$$

for the $USp(4)$ model. Specifically, these constraints are further restricted by various Bianchi identities for the superconnections ∇_I, and we treat all such relations to compute various derivatives of the superfields in our system. We will find that any higher derivatives of a superfield can be expressed by some other superfields and some lower derivatives of the superfields. Thus we obtain necessary and sufficient superfields and/or their derivatives to express all the other superfields and their derivatives, which corresponds to the independent degrees of freedom of the system. Since then all derivatives of all superfield are obtained, we can expand those superfields componentwisely. Thus we completely determine the explicit forms of the superfields, which we say we solve the constraints.

Let us now carry out the program explicitly. Most of the computations are based on Bianchi identities. For one simply Hermitian conjugate to the other only the result will be listed.

Three Fermionic Derivatives First we compute relations derived from various combinations of three fermionic derivatives. In order to show the typical manner of the computations, we list them in the full detail.

(i) Computation of $G_{i\alpha}$

$$
\begin{aligned}
G_{i\alpha} &= i[\nabla_{i\alpha}, \nabla_z] = i[\nabla_{i\alpha}, -\frac{i}{8}C^{\beta\gamma}\Omega^{jk}\{\nabla_{j\beta}, \nabla_{k\gamma}\}] \\
&= -\frac{1}{8}C^{\beta\gamma}\Omega^{jk}\Big([\nabla_{j\beta}, \{\nabla_{k\gamma}, \nabla_{i\alpha}\}] + [\nabla_{k\gamma}, \{\nabla_{i\alpha}, \nabla_{j\beta}\}]\Big) \\
&= -\frac{1}{8}C^{\beta\gamma}\Omega^{jk}\Big([\nabla_{j\beta}, i\Omega_{ki}C_{\gamma\alpha}\nabla_z - iC_{\gamma\alpha}W_{ki}] + [\nabla_{k\gamma}, i\Omega_{ij}C_{\alpha\beta}\nabla_z - iC_{\alpha\beta}W_{ij}]\Big) \\
&= -\frac{i}{4}\Big([\nabla_{i\alpha}, \nabla_z] + [\nabla_{j\alpha}, W^j{}_i]\Big) = -\frac{i}{4}\Big(-iG_{i\alpha} + [\nabla_{j\alpha}, W^j{}_i]\Big),
\end{aligned}
$$

so that

$$
[\nabla_{j\alpha}, W^j{}_i] = 5iG_{i\alpha}. \tag{C.1}
$$

(i′) Computation of $\overline{G}_{i\dot\alpha}$

$$
[\overline{\nabla}_{j\dot\alpha}, W^j{}_i] = 5i\overline{G}_{i\dot\alpha}, \tag{C.2}
$$

consistent to (i).

(ii) Another computation of $G_{i\alpha}$

$$
\begin{aligned}
G_{i\alpha} &= i[\nabla_{i\alpha}, \nabla_z] = i[\nabla_{i\alpha}, -\frac{i}{8}C^{\dot\beta\dot\gamma}\Omega^{jk}\{\overline{\nabla}_{j\dot\beta}, \overline{\nabla}_{k\dot\gamma}\}] \\
&= -\frac{1}{8}C^{\dot\beta\dot\gamma}\Omega^{jk}\Big([\overline{\nabla}_{j\dot\beta}, i\Omega_{ik}(\sigma^\mu)_{\alpha\dot\gamma}\nabla_\mu] + [\overline{\nabla}_{k\dot\gamma}, i\Omega_{ij}(\sigma^\mu)_{\alpha\dot\beta}\nabla_\mu]\Big) \\
&= \frac{i}{4}(\sigma^\mu C)_\alpha{}^{\dot\beta}[\overline{\nabla}_{i\dot\beta}, \nabla_\mu] = -\frac{1}{4}(\sigma^\mu C)_\alpha{}^{\dot\beta}\overline{F}_{i\dot\beta\mu},
\end{aligned}
$$

so that

$$
G_{i\alpha} = -\frac{1}{4}(\sigma^\mu C)_\alpha{}^{\dot\beta}\overline{F}_{i\dot\beta\mu}. \tag{C.3}
$$

(ii′) Another computation of $\overline{G}_{i\dot{\alpha}}$

$$\overline{G}_{i\dot{\alpha}} = -\frac{1}{4}(C\sigma^\mu)^\beta{}_{\dot{\alpha}} F_{i\beta\mu}, \tag{C.4}$$

consistent to (ii).

(iii) Computation of the first derivative of W_{jk}

$$\begin{aligned}
[\nabla_{i\alpha}, W_{jk}] &= [\nabla_{i\alpha}, \frac{i}{2}C^{\beta\gamma}\Big(\{\nabla_{j\beta}, \nabla_{k\gamma}\} - iC\beta\gamma\Omega_{jk}\nabla_z\Big)] \\
&= -\frac{i}{2}C^{\beta\gamma}\Big([\nabla_{j\beta}, i\Omega_{ki}C_{\gamma\alpha}\nabla_z - iC_{\gamma\alpha}W_{ki}] + [\nabla_{k\gamma}, i\Omega_{ij}C_{\alpha\beta}\nabla_z - iC_{\alpha\beta}W_{ij}]\Big) \\
&\quad + \Omega_{jk}[\nabla_{i\alpha}, \nabla_z] \\
&= -\frac{1}{2}\Big(\Omega_{ki}[\nabla_{j\alpha}, \nabla_z] + \Omega_{ij}[\nabla_{k\alpha}, \nabla_z]\Big) + \Omega_{jk}[\nabla_{i\alpha}, \nabla_z] \\
&\quad + \frac{1}{2}\Big([\nabla_{j\alpha}, W_{ki}] + [\nabla_{k\alpha}, W_{ij}]\Big),
\end{aligned}$$

so that

$$[\nabla_{i\alpha}, W_{ji}] = \frac{i}{2}\Omega_{i[j}G_{k]\alpha} - i\Omega_{jk}G_{i\alpha} + \frac{1}{2}[\nabla_{[j\alpha}, W_{k]i}].$$

(iii′) Computation of the first derivative of W_{jk}

$$[\overline{\nabla}_{i\dot{\alpha}}, W_{jk}] = \frac{i}{2}\Omega_{i[j}\overline{G}_{k]\dot{\alpha}} - i\Omega_{jk}\overline{G}_{i\dot{\alpha}} + \frac{1}{2}[\overline{\nabla}_{[j\dot{\alpha}}, W_{k]i}],$$

consistent to (iii).

(iv) Another computation of the first derivative of W_{jk}

$$\begin{aligned}
[\nabla_{i\alpha}, W_{jk}] &= [\nabla_{i\alpha}, -\frac{i}{2}C^{\dot{\beta}\dot{\gamma}}\Big(\{\overline{\nabla}_{j\dot{\beta}}, \overline{\nabla}_{k\dot{\gamma}}\} - iC_{\dot{\beta}\dot{\gamma}}\Omega_{jk}\nabla_z\Big)] \\
&= \frac{i}{2}C^{\dot{\beta}\dot{\gamma}}\Big([\overline{\nabla}_{j\dot{\beta}}, i\Omega_{ik}(\sigma^\mu)_{\alpha\dot{\gamma}}\nabla_\mu] + [\overline{\nabla}_{k\dot{\gamma}}, i\Omega_{ij}(\sigma^\mu)_{\alpha\dot{\beta}}\nabla_\mu]\Big) \\
&\quad - \Omega_{jk}[\nabla_{i\alpha}, \nabla_z] \\
&= -\frac{i}{2}\Omega_{i[j}(\sigma^\mu C)_\alpha{}^{\dot{\beta}}\overline{F}_{k]\dot{\beta}\mu} + i\Omega_{jk}G_{i\alpha},
\end{aligned}$$

so that

$$[\nabla_{i\alpha}, W_{jk}] = 2i\Omega_{i[j}G_{k]\alpha} + i\Omega_{jk}G_{i\alpha}. \tag{C.5}$$

This solves (iii) consistently.

(iv′) Another computation of the first derivative of W_{jk}

$$[\overline{\nabla}_{i\dot{\alpha}}, W_{jk}] = 2i\Omega_{i[j}\overline{G}_{k]\dot{\alpha}} + i\Omega_{jk}\overline{G}_{i\dot{\alpha}}, \tag{C.6}$$

consistent to (iv).

(v) Computation of $F_{i\alpha\mu}$

$$\begin{aligned}
F_{i\alpha\mu} &= i[\nabla_{i\alpha}, \nabla_\mu] = i[\nabla_{i\alpha}, -\frac{i}{8}\Omega^{jk}(\bar{\sigma}_\mu)^{\dot{\gamma}\beta}\{\nabla_{j\beta}, \overline{\nabla}_{k\dot{\gamma}}\}] \\
&= -\frac{1}{8}\Omega^{jk}(\bar{\sigma}_\mu)^{\dot{\gamma}\beta}\Big([\nabla_{j\beta}, i\Omega_{ik}(\sigma^\nu)_{\alpha\dot{\gamma}}\nabla_\nu] + [\overline{\nabla}_{k\dot{\gamma}}, i\Omega_{ij}C_{\alpha\beta}\nabla_z - iC_{\alpha\beta}W_{ij}]\Big) \\
&= -\frac{1}{8}(\sigma^\mu\bar{\sigma}_\mu)_\alpha{}^\beta F_{i\beta\nu} + \frac{3}{4}(\bar{\sigma}_\mu C)^{\dot{\gamma}}{}_\alpha\overline{G}_{i\dot{\gamma}}
\end{aligned}$$

$$= -\frac{1}{4}F_{i\alpha\mu} - \frac{1}{8}(\bar{\sigma}_\mu C)^{\dot{\gamma}}{}_\alpha (C\sigma^\nu)^\beta{}_{\dot{\gamma}} F_{i\beta\nu} + \frac{3}{4}(\bar{\sigma}_\mu C)^{\dot{\gamma}}{}_\alpha \overline{G}_{i\dot{\gamma}},$$

so using eq. (C.4), we obtain

$$F_{i\alpha\mu} = (\bar{\sigma}_\mu C)^{\dot{\gamma}}{}_\alpha \overline{G}_{i\dot{\gamma}}. \tag{C.7}$$

(v′) Computation of $\overline{F}_{i\dot{\alpha}\mu}$

$$\overline{F}_{i\dot{\alpha}\mu} = (C\bar{\sigma}_\mu)_{\dot{\alpha}}{}^\beta G_{i\beta}, \tag{C.8}$$

consistent to (v).

Four Fermionic Derivatives We then show relations constructed by four fermionic derivatives. Those computations tend to very complicated. Since we have seen in the preceding paragraph the typical rules for such computations, we now list the process more simply in the following.

(vi) Computation of the first derivative of $G_{j\beta}$

$$\begin{aligned}\{\nabla_{i\alpha}, G_{j\beta}\} &= i\{\nabla_{i\alpha}, [\nabla_{j\beta}, \nabla_z]\}\\ &= -\{\nabla_{j\beta}, G_{i\alpha}\} + [\nabla_z, \Omega_{ij}C_{\alpha\beta}\nabla_z - C_{\alpha\beta}W_{ij}],\end{aligned}$$

so that

$$\{\nabla_{i\alpha}, G_{j\beta}\} + \{\nabla_{j\beta}, G_{i\alpha}\} = -C_{\alpha\beta}[\nabla_z, W_{ij}].$$

(vi′) Computation of the first derivative of $\overline{G}_{j\dot{\beta}}$

$$\{\overline{\nabla}_{i\dot{\alpha}}, \overline{G}_{j\dot{\beta}}\} + \{\overline{\nabla}_{j\dot{\beta}}, \overline{G}_{i\dot{\alpha}}\} = -C_{\dot{\alpha}\dot{\beta}}[\nabla_z, W_{ij}].$$

(vii) Another computation of the first derivative of $G_{i\beta}$

$$\begin{aligned}\{\nabla_{i\alpha}, G_{j\beta}\} &= -\frac{i}{5}\{\nabla_{i\alpha}, [\nabla_{k\beta}, W^k{}_j]\}\\ &= \frac{1}{5}\Big(\{\nabla_{j\beta}, G_{i\alpha}\} - 2\{\nabla_{i\beta}, G_{j\alpha}\} + 2\Omega_{ij}\{\nabla_{k\beta}, G^k{}_\alpha\}\\ &\qquad - C_{\alpha\beta}[\nabla_z, W_{ij}] + C_{\alpha\beta}[W^k{}_j, W_{ik}]\Big).\end{aligned}$$

Taking (i,j), we find that

$$\{\nabla_{(i\alpha}, G_{j)\beta}\} = -\frac{1}{2}[W_{ik}, W^k{}_j], \tag{C.9}$$

and, taking $[i,j]$, that

$$\{\nabla_{[i\alpha}, G_{j]\beta}\} = -C_{\alpha\beta}[\nabla_z, W_{ij}] + \frac{1}{2}\Omega_{ij}\{\nabla_{k\alpha}, G^k{}_\beta\}.$$

(vii′) Another computation of the first derivative of $\overline{G}_{j\dot{\beta}}$

$$\begin{aligned}\{\overline{\nabla}_{(i\dot{\alpha}}, \overline{G}_{j)\dot{\beta}}\} &= \frac{1}{2}C_{\dot{\alpha}\dot{\beta}}[W_{ik}, W^k{}_j], \\ \{\overline{\nabla}_{[i\dot{\alpha}}, \overline{G}_{j]\dot{\beta}}\} &= -C_{\dot{\alpha}\dot{\beta}}[\nabla_z, W_{ij}] + \frac{1}{2}\Omega_{ij}\{\overline{\nabla}_{k\dot{\alpha}}, \overline{G}^k{}_{\dot{\beta}}\}.\end{aligned} \tag{C.10}$$

(viii) Further computation of the first derivative of $G_{j\beta}$

$$\{\nabla_{i\alpha}, G_{j\beta}\} = -\frac{1}{4}(\sigma^\mu C)_\beta{}^{\dot{\beta}}\{\nabla_{i\alpha}, \overline{F}_{j\dot{\beta}\mu}\} = \frac{i}{4}(\sigma^\mu C)_\beta{}^{\dot{\beta}}\{\nabla_{i\alpha}, [\overline{\nabla}_{j\dot{\beta}}, \nabla_\mu]\}$$

$$= -\frac{1}{2}C_{\alpha\beta}\{\overline{\nabla}_{j\dot{\alpha}}, \overline{G}_i{}^{\dot{\alpha}}\} - \frac{i}{4}\Omega_{ij}(\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu}.$$

We then find that using (vi′)

$$\{\nabla_{i\alpha}, G^i{}_\beta\} = \frac{1}{2}C_{\alpha\beta}\{\overline{\nabla}_{i\dot{\alpha}}, \overline{G}^{i\dot{\alpha}}\} - i(\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu} = -i(\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu},$$

which, together with (vii), leads to

$$\{\nabla_{[i\alpha}, G_{j]\beta}\} = -C_{\alpha\beta}[\nabla_z, W_{ij}] - \frac{i}{2}\Omega_{ij}(\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu}. \tag{C.11}$$

Thus, from eqs. (C.9) and (C.11), we obtain that

$$\{\nabla_{i\alpha}, G_{j\beta}\} = -\frac{i}{4}\Omega_{ij}(\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu} - \frac{1}{2}C_{\alpha\beta}[\nabla_z, W_{ij}] - \frac{1}{4}C_{\alpha\beta}[W_{ik}, W^k{}_j]. \tag{C.12}$$

(viii′) Further computation of the first derivative of $\overline{G}_{j\dot{\beta}}$

$$\{\overline{\nabla}_{[i\dot{\alpha}}, \overline{G}_{j]\dot{\beta}}\} = -C_{\dot{\alpha}\dot{\beta}}[\nabla_z, W_{ij}] - \frac{i}{2}\Omega_{ij}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}F_{\mu\nu}, \tag{C.13}$$

$$\{\overline{\nabla}_{i\dot{\alpha}}, \overline{G}_{j\dot{\beta}}\} = -\frac{i}{4}\Omega_{ij}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}F_{\mu\nu} - \frac{1}{2}C_{\dot{\alpha}\dot{\beta}}[\nabla_z, W_{ij}] + \frac{1}{4}C_{\dot{\alpha}\dot{\beta}}[W_{ik}, W^k{}_j]. \tag{C.14}$$

(ix) Computation of the first derivative of $G_{j\beta}$

$$\begin{aligned}\{\overline{\nabla}_{i\dot{\alpha}}, G_{j\beta}\} &= i\{\overline{\nabla}_{i\dot{\alpha}}, [\nabla_{j\beta}, \nabla_z]\}\\ &= -i\Big(i\{\nabla_{j\beta}, \overline{G}_{i\dot{\alpha}}\} + i\Omega_{ji}(\sigma^\mu)_{\beta\dot{\alpha}}[\nabla_z, \nabla_\mu]\Big) = \{\nabla_{j\beta}, \overline{G}_{i\dot{\alpha}}\} - i\Omega_{ij}(\sigma^\mu)_{\beta\dot{\alpha}}g_\mu.\end{aligned}$$

(ix′) Computation of the first derivative of $\overline{G}_{j\dot{\beta}}$

$$\{\nabla_{i\alpha}, \overline{G}_{j\dot{\beta}}\} = \{\overline{\nabla}_{j\dot{\beta}}, G_{i\alpha}\} - i\Omega_{ij}(\sigma^\mu)_{\alpha\dot{\beta}}g_\mu.$$

(x) Another computation of the first derivative of $G_{j\beta}$

$$\begin{aligned}\{\overline{\nabla}_{i\dot{\alpha}}, G_{j\beta}\} &= -\frac{i}{5}\{\overline{\nabla}_{i\dot{\alpha}}, [\nabla_{k\beta}, W^k{}_j]\}\\ &= \frac{1}{5}\Big(\{\nabla_{j\beta}, \overline{G}_{i\dot{\alpha}}\} - 2\{\nabla_{i\beta}, \overline{G}_{j\dot{\alpha}}\} + 2\Omega_{ij}\{\nabla_{k\beta}, \overline{G}^k{}_{\dot{\alpha}}\}\Big) + \frac{1}{5}(\sigma^\mu)_{\beta\dot{\alpha}}[\nabla_\mu, W_{ij}].\end{aligned}$$

Taking (i, j) we find, with the use of (ix′), that

$$\{\overline{\nabla}_{(i\dot{\alpha}}, G_{j)\beta}\} = 0, \tag{C.15}$$

and, similarly, $[i, j]$,

$$\{\overline{\nabla}_{[i\dot{\alpha}}, G_{j]\beta}\} = -5i\Omega_{ij}(\sigma^\mu)_{\beta\dot{\alpha}}g_\mu - 2\Omega_{ij}\{\overline{\nabla}_{k\dot{\alpha}}, G^k{}_\beta\} + (\sigma^\mu)_{\beta\dot{\alpha}}[\nabla_\mu, W_{ij}].$$

Multiplying the last equation by Ω^{ij}, we have

$$\{\overline{\nabla}_{k\dot{\alpha}}, G^k{}_\beta\} = -2i(\sigma^\mu)_{\beta\dot{\alpha}}g_\mu,$$

which in turn leads us to

$$\{\overline{\nabla}_{[i\dot{\alpha}}, G_{j]\beta}\} = -i\Omega_{ij}(\sigma^\mu)_{\beta\dot{\alpha}}g_\mu + (\sigma^\mu)_{\beta\dot{\alpha}}[\nabla_\mu, W_{ij}]. \tag{C.16}$$

Thus using eqs. (C.15) and (C.16), we obtain that

$$\{\overline{\nabla}_{i\dot{\alpha}}, G_{j\beta}\} = -\frac{1}{2}(\sigma^\mu)_{\beta\dot{\alpha}}\Big(i\Omega_{ij}g_\mu - [\nabla_\mu, W_{ij}]\Big). \tag{C.17}$$

(x′) Another computation of the first derivative of $\overline{G}_{j\dot\beta}$

$$\{\nabla_{(i\alpha}, \overline{G}_{j)\dot\beta}\} = 0, \tag{C.18}$$

$$\{\nabla_{[i\alpha}, \overline{G}_{j]\dot\beta}\} = -i\Omega_{ij}(\sigma^\mu)_{\alpha\dot\beta}g_\mu - (\sigma^\mu)_{\alpha\dot\beta}[\nabla_\mu, W_{ij}], \tag{C.19}$$

$$\{\nabla_{i\alpha}, \overline{G}_{j\dot\beta}\} = -\frac{1}{2}(\sigma^\mu)_{\alpha\dot\beta}\Big(i\Omega_{ij}g_\mu + [\nabla_\mu, W_{ij}]\Big). \tag{C.20}$$

(xi) Computation of g_μ

$$g_\mu = i[\nabla_\mu, \nabla_z] = \frac{1}{8}(\bar\sigma^\mu)^{\dot\beta\alpha}[\{\nabla_{i\alpha}, \overline{\nabla}^i{}_{\dot\beta}\}, \nabla_z] = \frac{i}{8}(\bar\sigma^\mu)^{\dot\beta\alpha}\Big(\{\nabla_{i\alpha}, \overline{G}^i{}_{\dot\beta}\} + \{\overline{\nabla}_{i\dot\beta}, G^i_\alpha\}\Big).$$

(xii) Another computation of g_μ

$$\begin{aligned} g_\mu = i[\nabla_\mu, \nabla_z] &= \frac{1}{8}\Omega^{ij}C^{\alpha\beta}[\nabla_\mu, \{\nabla_{i\alpha}, \nabla_{j\beta}\}] \\ &= \frac{i}{4}(\bar\sigma_\mu)^{\dot\beta\alpha}\{\nabla_{i\alpha}, \overline{G}^i{}_{\dot\beta}\}. \end{aligned}$$

(xii′) Another computation of g_μ

$$g_\mu = \frac{i}{4}(\bar\sigma_\mu)^{\dot\beta\alpha}\{\overline{\nabla}_{i\dot\beta}, G^i{}_\alpha\}.$$

These results ((xi), (xii), (xii′)) are trivially obtainable from (x) and (x′).

Five Fermionic Derivatives We now go on to the computation of relations containing five fermionic derivatives.

(xiii) Computation of $[\nabla_{i\alpha}, g_\mu]$

$$[\nabla_{i\alpha}, g_\mu] = i[\nabla_{i\alpha}, [\nabla_\mu, \nabla_z]] = [\nabla_\mu, G_{i\alpha}] - (\bar\sigma_\mu C)^{\dot\alpha}{}_\alpha[\nabla_z, \overline{G}_{i\dot\alpha}].$$

(xiii′) Computation of $[\overline{\nabla}_{i\dot\alpha}, g_\mu]$

$$[\overline{\nabla}_{i\dot\alpha}, g_\mu] = -[\nabla_\mu, \overline{G}_{i\dot\alpha}] + (C\bar\sigma_\mu)_{\dot\alpha}{}^\alpha[\nabla_z, G_{i\alpha}].$$

(xiv) Another computation $[\nabla_{i\alpha}, g_\mu]$

$$[\nabla_{i\alpha}, g_\mu] = \frac{i}{4}(\bar\sigma^\mu)^{\dot\beta\beta}[\nabla_{i\alpha}, \{\nabla_{j\beta}, \overline{G}^j{}_{\dot\beta}\}],$$

where

$$\begin{aligned} [\nabla_{i\alpha}, \{\nabla_{j\beta}, \overline{G}^j{}_{\dot\beta}\}] = [\nabla^j{}_\beta, -\frac{1}{2}(\sigma^\mu)_{\alpha\dot\beta}\Big(i\Omega_{ij}g_\mu + [\nabla_\mu, W_{ij}]\Big)] - [\overline{G}^j{}_{\dot\beta}, i\Omega_{ij}C_{\alpha\beta}\nabla_z - iC_{\alpha\beta}W_{ij}] \\ = \frac{i}{2}(\sigma^\mu)_{\alpha\dot\beta}[\nabla_{i\beta}, g_\mu] - \frac{5}{2}i(\sigma^\mu)_{\alpha\dot\beta}[\nabla_\mu, G_{i\beta}] \\ + 2iC_{\alpha\beta}[\overline{G}^j{}_{\dot\beta}, W_{ij}] - iC_{\alpha\beta}[\nabla_z, \overline{G}_{i\dot\beta}]. \end{aligned}$$

Thus we find that

$$\begin{aligned} [\nabla_{i\alpha}, g_\mu] = -\frac{1}{8}(\sigma^\nu\bar\sigma_\mu)_\alpha{}^\beta[\nabla_{i\beta}, g_\nu] - \frac{5}{8}(\sigma^\nu\bar\sigma_\mu)_\alpha{}^\beta[\nabla_\nu, G_{i\beta}] \\ + \frac{1}{2}(\bar\sigma_\mu C)^{\dot\beta}{}_\alpha[\overline{G}^j{}_{\dot\beta}, W_{ij}] - \frac{1}{4}(\bar\sigma_\mu C)^{\dot\beta}{}_\alpha[\nabla_z, \overline{G}_{i\dot\beta}]. \end{aligned}$$

(xiv′) Another computation of $[\overline{\nabla}_{i\dot{\alpha}}, g_\mu]$

$$[\overline{\nabla}_{i\dot{\alpha}}, g_\mu] = -\frac{1}{8}(\bar{\sigma}_\mu\sigma^\nu)^{\dot{\beta}}{}_{\dot{\alpha}}[\overline{\nabla}_{i\dot{\beta}}, g_\nu] - \frac{5}{8}(\bar{\sigma}_\mu\sigma_\nu)^{\dot{\beta}}{}_{\dot{\alpha}}[\nabla^\nu, \overline{G}_{i\dot{\beta}}] + \frac{1}{2}(C\bar{\sigma}_\mu)_{\dot{\alpha}}{}^\beta[G^j{}_\beta, W_{ij}] + \frac{1}{4}(C\bar{\sigma}_\mu)_{\dot{\alpha}}{}^\beta[\nabla_z, G_{i\beta}].$$

(xv) Further computation of $[\nabla_{i\alpha}, g_\mu]$

$$[\nabla_{i\alpha}, g_\mu] = \frac{i}{4}(\bar{\sigma})^{\dot{\beta}\beta}[\nabla_{i\alpha}, \{\overline{\nabla}_{j\dot{\beta}}, G^j{}_\beta\}].$$

Since

$$\begin{aligned}
[\nabla_{i\alpha}, \{\overline{\nabla}_{j\dot{\beta}}, G^j{}_\beta\}] &= \left[\overline{\nabla}^j{}_{\dot{\beta}}, -\frac{i}{4}\Omega_{ij}(\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu} - \frac{1}{2}C_{\alpha\beta}[\nabla_z, W_{ij}] - \frac{1}{4}C_{\alpha\beta}[W_{ik}, W^k{}_j]\right] \\
&\quad + i\Omega_{ij}(\sigma^\nu)_{\alpha\dot{\beta}}[\nabla_\nu, G^j{}_\beta], \\
&\quad \left|\; \begin{pmatrix} (\sigma^{\mu\nu})_{\alpha\beta}[\overline{\nabla}_{i\dot{\beta}}, F_{\mu\nu}] = i(\sigma^{\mu\nu})_{\alpha\beta}[\overline{\nabla}_{i\dot{\beta}}, [\nabla_\mu, \nabla_\nu]] \\ = -2\big(C_{\alpha\beta}(C\bar{\sigma}^\mu)^{\dot{\gamma}\gamma}[\nabla_\mu, G_{i\gamma}] + 2(\sigma^\mu)_{\beta\dot{\beta}}[\nabla_{\mu, G_{i\alpha}}]\big), \\ [\overline{\nabla}^j{}_{\dot{\beta}}, [\nabla_z, W_ij]] = 5i[\nabla_z, \overline{G}_{i\dot{\beta}}] + i[\overline{G}^j{}_{\dot{\beta}}, W_{ij}], \\ [\overline{\nabla}^j{}_{\dot{\beta}}, [W_{ik}, W^k{}_j]] = -5i[\overline{G}^k{}_{\dot{\beta}}, W_{ik}] - [i\Omega_{ik}\overline{G}_{j\dot{\beta}} + 2i\Omega_{j[i}\overline{G}_{k]\dot{\beta}}, W^{kj}] \\ = -8i[\overline{G}^k{}_{\dot{\beta}}, W_{ik}], \end{pmatrix} \right. \\
&= -\frac{i}{2}C_{\alpha\beta}(C\bar{\sigma}^\mu)_{\dot{\beta}}{}^\gamma[\nabla_\mu, G_{i\gamma}] - i(\sigma^\mu)_{\beta\dot{\beta}}[\nabla_\mu, G_{i\alpha}] - i(\sigma^\mu)_{\alpha\dot{\beta}}[\nabla_\mu, G_{i\beta}] \\
&\quad - \frac{5}{2}iC_{\alpha\beta}[\nabla_z, \overline{G}_{i\dot{\beta}}] + \frac{3}{2}iC_{\alpha\beta}[\overline{G}^j{}_{\dot{\beta}}, W_{ij}],
\end{aligned}$$

we find that

$$\begin{aligned}
[\nabla_{i\alpha}, g_\mu] &= \frac{5}{8}\left([\nabla_\mu, G_{i\alpha}] - (\bar{\sigma}_\mu C)^{\dot{\beta}}{}_\alpha[\nabla_z, \overline{G}_{i\dot{\beta}}]\right) \\
&\quad - \frac{3}{8}\left((\sigma_{\mu\nu})_\alpha{}^\beta[\nabla^\nu, G_{i\beta}] - (\bar{\sigma}_\mu C)^{\dot{\beta}}{}_\alpha[\overline{G}^j{}_{\dot{\beta}}, W_{ij}]\right). \qquad (*)
\end{aligned}$$

Equating this to the result in (xiv), we have

$$(\sigma^\nu\bar{\sigma}_\mu)_\alpha{}^\beta[\nabla_{i\beta}, g_\nu] = -2(\sigma_{\mu\nu})_\alpha{}^\beta[\nabla^\nu, G_{i\beta}] + (\bar{\sigma}_\mu C)^{\dot{\beta}}{}_\alpha[\overline{G}^j{}_{\dot{\beta}}, W_{ij}] + 3(\bar{\sigma}_\mu C)^{\dot{\beta}}{}_\alpha[\nabla_z, \overline{G}_{i\dot{\beta}}].$$

After multiply this by $\frac{1}{4}(\sigma^\mu\bar{\sigma}_\rho)_\delta{}^\alpha$, rename ρ as μ to give

$$\begin{aligned}
[\nabla_{i\alpha}, g_\mu] &= \frac{1}{2}\left((\sigma_{\mu\nu})_\alpha{}^\beta[\nabla^\nu, G_{i\beta}] - (\bar{\sigma}_\mu C)^{\dot{\beta}}{}_\alpha[\overline{G}^j{}_{\dot{\beta}}, W_{ij}]\right) \\
&\quad + \frac{3}{2}\left([\nabla_\mu, G_{i\alpha}] - (\bar{\sigma}_\mu C)^{\dot{\beta}}{}_\alpha[\nabla_z, \overline{G}_{i\dot{\beta}}]\right).
\end{aligned}$$

Equating this to $(*)$, we find that

$$[\nabla_\mu, G_{i\alpha}] - (\bar{\sigma}_\mu C)^{\dot{\beta}}{}_\alpha[\nabla_z, \overline{G}_{i\dot{\beta}}] = -(\sigma_{\mu\nu})_\alpha{}^\beta[\nabla^\nu, G_{i\beta}] + (\bar{\sigma}_\mu C)^{\dot{\beta}}{}_\alpha[\overline{G}^j{}_{\dot{\beta}}, W_{ij}].$$

Thus going back again to $(*)$, we obtain

$$\begin{aligned}
[\nabla_{i\alpha}, g_\mu] &= [\nabla_\mu, G_{i\alpha}] - (\bar{\sigma}_\mu C)^{\dot{\beta}}{}_\alpha[\nabla_z, \overline{G}_{i\dot{\beta}}] \\
&= -(\sigma_{\mu\nu})_\alpha{}^\beta[\nabla^\nu, G_{i\beta}] + (\bar{\sigma}_\mu C)^{\dot{\beta}}{}_\alpha[\overline{G}^j{}_{\dot{\beta}}, W_{ij}], \qquad \text{(C.21)}
\end{aligned}$$

which is completely consistent to (xiii). This result also leads to

$$[\nabla_z, \overline{G}_{i\dot{\alpha}}] = -(C\sigma^\mu)^\alpha{}_{\dot{\alpha}}[\nabla_\mu, G_{i\alpha}] - [\overline{G}^j{}_{\dot{\alpha}}, W_{ij}]. \qquad \text{(C.22)}$$

(xv′) Further computation of $[\overline{\nabla}_{i\dot{\alpha}}, g_\mu]$

$$[\overline{\nabla}_{i\dot{\alpha}}, g_\mu] = -\frac{5}{8}\left([\nabla_\mu, \overline{G}_{i\dot{\alpha}}] - (C\bar{\sigma}_\mu)_{\dot{\alpha}}{}^{\beta}[\nabla_z, G_{i\beta}]\right)$$
$$- \frac{3}{8}\left((\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}[\nabla^\nu, \overline{G}_{i\dot{\beta}}] - (C\bar{\sigma}_\mu)_{\dot{\alpha}}{}^{\beta}[G^j{}_\beta, W_{ij}]\right), \tag{$*'$}$$

$$[\overline{\nabla}_{i\dot{\alpha}}, g_\mu] = -[\nabla_\mu, \overline{G}_{i\dot{\alpha}}] + (C\bar{\sigma}_\mu)_{\dot{\alpha}}{}^{\beta}[\nabla_z, G_{i\beta}]$$
$$= -(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}_{\dot{\alpha}}[\nabla^\nu, \overline{G}_{i\dot{\beta}}] + (C\bar{\sigma}_\mu)_{\dot{\alpha}}{}^{\beta}[G^j{}_\beta, W_{ij}], \tag{C.23}$$

$$[\nabla_z, G_{i\alpha}] = -(\sigma^\mu C)_\alpha{}^{\dot{\beta}}[\nabla_\mu, \overline{G}_{i\dot{\beta}}] + [G^j{}_\alpha, W_{ij}]. \tag{C.24}$$

(xvi) Computation of $[\nabla_{i\alpha}, [\nabla_z, W_{jk}]]$

$$[\nabla_{i\alpha}, [\nabla_z, W_{jk}]] = i[\nabla_z, \Omega_{jk}G_{i\alpha} + 2\Omega_{i[j}G_{k]\alpha}] - i[G_{i\alpha}, W_{jk}]$$
$$= -i(\sigma^\mu C)_\alpha{}^{\dot{\alpha}}\left(\Omega_{jk}[\nabla_\mu, \overline{G}_{i\dot{\alpha}}] + 2\Omega_{i[j}[\nabla_\mu, \overline{G}_{k]\dot{\alpha}}]\right)$$
$$+ i\left(\Omega_{jk}[G^l{}_\alpha, W_{il}] + 2\Omega_{i[j}[G^l{}_\alpha, W_{k]l}]\right)$$
$$- i[G_{i\alpha}, W_{jk}]. \tag{C.25}$$

(xvii) Computation of $[\overline{\nabla}_{i\dot{\alpha}}, [\nabla_z, W_{jk}]]$

$$[\overline{\nabla}_{i\dot{\alpha}}, [\nabla_z, W_{jk}]] = -i(C\sigma^\mu)^\alpha{}_{\dot{\alpha}}\left(\Omega_{jk}[\nabla_\mu, G_{i\alpha}] + 2\Omega_{i[j}[\nabla_\mu, G_{k]\alpha}]\right)$$
$$- i\left(\Omega_{jk}[\overline{G}^l{}_{\dot{\alpha}}, W_{il}] + 2\Omega_{i[j}[\overline{G}^l{}_{\dot{\alpha}}, W_{k]l}]\right)$$
$$+ i[\overline{G}_{i\dot{\alpha}}, W_{jk}].$$

Six Fermionic Derivatives Finally we compute relations containing six fermionic derivatives.

(xviii) Computation of $[\nabla_z, [\nabla_z, W_{jk}]]$

$$[\nabla_z, [\nabla_z, W_{jk}]] = -\frac{i}{8}\left[\{\nabla_{i\alpha}, \nabla^{i\alpha}\}, [\nabla_z, W_{jk}]\right] = -\frac{i}{4}\{\nabla_{i\alpha}, [\nabla_{i\alpha}, [\nabla_z, W_{jk}]]\}$$
$$= -\frac{1}{4}\Big\{\nabla^{i\alpha}, (\sigma^\mu C)_\alpha{}^{\dot{\alpha}}\left(\Omega_{jk}[\nabla_\mu, \overline{G}_{i\dot{\alpha}}] + 2\Omega_{i[j}[\nabla_\mu, \overline{G}_{k]\dot{\alpha}}]\right)$$
$$- \left(\Omega_{jk}[G^j{}_\alpha, W_{il}] + 2\Omega_{i[j}[G^l{}_\alpha, W_{k]l}]\right) + [G_{i\alpha}, W_{jk}]\Big\}$$

$$\left|\;\left(\begin{aligned}
\{\nabla^{i\alpha}, [\nabla_\mu, \overline{G}_{i\dot{\alpha}}]\} &= 2i(C\sigma^\nu)^\alpha{}_{\dot{\alpha}}[\nabla_\mu, g_\nu] - i(\bar{\sigma}_\mu)^{\dot{\beta}\alpha}\{\overline{G}_{i\dot{\alpha}}, \overline{G}^i{}_{\dot{\alpha}}\},\\
\{\nabla^{i\alpha}, [\nabla_\mu, \overline{G}_{k]\dot{\alpha}}]\} &= -\frac{1}{2}(C\sigma^\nu)^\alpha{}_{\dot{\alpha}}\left(-i\delta_{k]}{}^i[\nabla_\mu, g_\nu] + [\nabla_\mu, [\nabla_\nu, W^i{}_{k]}]]\right)\\
&\quad - i(\bar{\sigma}_\mu)^{\dot{\beta}\alpha}\{\overline{G}_{k]\dot{\alpha}}, \overline{G}^i{}_{\dot{\beta}}\},\\
\{\nabla^{i\alpha}, [G^l{}_\alpha, W_{il}]\} &= 5i\{G^l{}_\alpha, G_l{}^\alpha\} - [W_{il}, [\nabla_z, W^{il}]],\\
\{\nabla^{i\alpha}, [G^l{}_\alpha, W_{k]l}]\} &= 3i\{G_{k]\alpha}, G^{i\alpha}\} - 2i\delta^i{}_{k]}\{G_{l\alpha}, G^{l\alpha}\}\\
&\quad - [W_{k]l}, [\nabla_z, W^{il}]] - \frac{1}{2}[W_{k]l}, [W^i{}_m, W^{ml}]],\\
\{\nabla^{i\alpha}, [G_{i\alpha}, W_{jk}]\} &= -i\Omega_{jk}\{G_{i\alpha}, G^{i\alpha}\} + 4i\{G_{j\alpha}, G_k{}^\alpha\},
\end{aligned}\right)\right.$$

$$= [\nabla_\mu, [\nabla_\mu, W_{jk}]] + \frac{1}{4}[W_{[j|l}, [W_{k]m}, W^{ml}]]$$
$$+ i\left(\Omega_{jk}\{G_{i\alpha}, G^{i\alpha}\} - 4\{G_{j\alpha}, G_k{}^\alpha\}\right) - i\left(\Omega_{jk}\{\overline{G}_{i\dot{\alpha}}, \overline{G}^{i\dot{\alpha}}\} - 4\{\overline{G}_{j\dot{\alpha}}, \overline{G}_k{}^{\dot{\alpha}}\}\right)$$
$$- \frac{1}{4}\left(\Omega_{jk}[W_{il}, [\nabla_z, W^{il}]] - 2[W_{[j|l}, [\nabla_z, W_{k]}{}^l]]\right).$$

(xviii′) Another computation of $[\nabla_z, [\nabla_z, W_{jk}]]$

$$
\begin{aligned}
[\nabla_z, [\nabla_z, W_{jk}]] &= -\frac{i}{8}\left[\{\overline{\nabla}_{i\dot\alpha}, \overline{\nabla}^{i\dot\alpha}\}, [\nabla_z, W_{jk}]\right] = -\frac{i}{4}\left\{\overline{\nabla}_{i\dot\alpha}, [\overline{\nabla}_{i\dot\alpha}, [\nabla_z, W_{jk}]]\right\} \\
&= [\nabla_\mu, [\nabla_\mu, W_{jk}]] + \frac{1}{4}[W_{[j|l}, [W_{k]m}, W^{ml}]] \\
&\quad + i\left(\Omega_{jk}\{G_{i\alpha}, G^{i\alpha}\} - 4\{G_{j\alpha}, G_k{}^\alpha\}\right) - i\left(\Omega_{jk}\{\overline{G}_{i\dot\alpha}, \overline{G}^{i\dot\alpha}\} - 4\{\overline{G}_{j\dot\alpha}, \overline{G}_k{}^{\dot\alpha}\}\right) \\
&\quad + \frac{1}{4}\left(\Omega_{jk}[W_{il}, [\nabla_z, W^{il}]] - 2[W_{[j|l}, [\nabla_z, W_{k]}{}^l]]\right).
\end{aligned}
$$

Comparing this result with the one in (xviii), we find immediately that we have to set

$$
\Omega_{jk}[W_{il}, [\nabla_z, W^{il}]] - 2[W_{[j|l}, [\nabla_z, W_{k]}{}^l]] = 0.
$$

Thus we obtain that

$$
\begin{aligned}
[\nabla_z, [\nabla_z, W_{jk}]] &= [\nabla_\mu, [\nabla_\mu, W_{jk}]] + \frac{1}{4}[W_{[j|l}, [W_{k]m}, W^{ml}]] \\
&\quad + i\left(\Omega_{jk}\{G_{i\alpha}, G^{i\alpha}\} - 4\{G_{j\alpha}, G_k{}^\alpha\}\right) - i\left(\Omega_{jk}\{\overline{G}_{i\dot\alpha}, \overline{G}^{i\dot\alpha}\} - 4\{\overline{G}_{j\dot\alpha}, \overline{G}_k{}^{\dot\alpha}\}\right).
\end{aligned}
\tag{C.26}
$$

(xix) Computation of $[\nabla_z, g_\mu]$

$$
\begin{aligned}
[\nabla_z, g_\mu] &= \frac{i}{4}(\bar\sigma^\mu)^{\dot\alpha\alpha}[\nabla_z, \{\nabla_{i\alpha}, \overline{G}^i{}_{\dot\alpha}\}] \\
&= \frac{i}{4}(\bar\sigma^\mu)^{\dot\alpha\alpha}\Big(\{\nabla_{i\alpha}, -(C\sigma^\nu)^\beta{}_{\dot\beta}[\nabla_\nu, G^i{}_\beta] - [\overline{G}^j{}_{\dot\alpha}, W^i{}_j]\} + i\{\overline{G}^j{}_{\dot\alpha}, G_{i\alpha}\}\Big) \\
&\Bigg|\ \left(\begin{aligned} \{\nabla_{i\alpha}, [\nabla_\nu, G^i{}_\beta]\} &= -i(\sigma^{\rho\sigma})_{\alpha\beta}[\nabla_\nu, F_{\rho\sigma}] - i(\bar\sigma_\nu C)^{\dot\beta}{}_\alpha\{G^i{}_\beta, \overline{G}_{i\dot\beta}\}, \\ \{\nabla_{i\alpha}, [\overline{G}^j{}_{\dot\alpha}, W^i{}_j]\} &= -5i\{\overline{G}^j{}_{\dot\alpha}, G_{j\alpha}\} + \frac{1}{2}(\sigma^\nu)_{\alpha\dot\alpha}[W^i{}_j, [\nabla_\nu, W_i{}^j]], \end{aligned}\right) \\
&= \wp^+{}_\mu{}^{\nu\rho\sigma}[\nabla_\nu, F_{\rho\sigma}] - 2(\bar\sigma_\mu)^{\dot\alpha\alpha}\{G_{i\alpha}, \overline{G}^i{}_{\dot\alpha}\} + \frac{i}{4}[W^{ij}, [\nabla_\nu, W_{ij}]].
\end{aligned}
\tag{$*$}
$$

Using the Bianchi identity w.r.t. $(\nabla_\nu, \nabla_\rho, \nabla_\sigma)$, i.e.,

$$
\varepsilon^{\mu\nu\rho\sigma}[\nabla_\nu, [\nabla_\rho, \nabla_\sigma]] = 0,
$$

the first term in $(*)$ equals simply to $[\nabla^\nu, F_{\mu\nu}]$. Thus we obtain

$$
[\nabla_z, g_\mu] = [\nabla^\nu, F_{\mu\nu}] - 2(\bar\sigma_\mu)^{\dot\alpha\alpha}\{G_{i\alpha}, \overline{G}^i{}_{\dot\alpha}\} + \frac{i}{4}[W^{ij}, [\nabla_\nu, W_{ij}]]. \tag{C.27}
$$

(xix′) Another computation of $[\nabla_z, g_\mu]$

$$
\begin{aligned}
[\nabla_z, g_\mu] &= \wp^-{}_\mu{}^{\nu\rho\sigma}[\nabla_\nu, F_{\rho\sigma}] - 2(\bar\sigma_\mu)^{\dot\alpha\alpha}\{G_{i\alpha}, \overline{G}^i{}_{\dot\alpha}\} + \frac{i}{4}[W^{ij}, [\nabla_\nu, W_{ij}]] \qquad (*') \\
&= [\nabla^\nu, F_{\mu\nu}] - 2(\bar\sigma_\mu)^{\dot\alpha\alpha}\{G_{i\alpha}, \overline{G}^i{}_{\dot\alpha}\} + \frac{i}{4}[W^{ij}, [\nabla_\nu, W_{ij}]].
\end{aligned}
$$

(xx) Computation of $[\nabla^\mu, g_\mu]$

$$
\begin{aligned}
[\nabla^\mu, g_\mu] &= -\frac{i}{8}(\bar\sigma^\mu)^{\dot\alpha\alpha}[\{\nabla_{i\alpha}, \overline{\nabla}^i{}_{\dot\alpha}\}, g_\mu] \\
&= -\frac{i}{8}(\bar\sigma^\mu)^{\dot\alpha\alpha}\Big(\{-(\bar\sigma_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}[\nabla^\nu, \overline{G}^i{}_{\dot\beta}] + (C\bar\sigma_\mu)_{\dot\alpha}{}^\beta[G^j{}_\beta, W^i{}_j], \nabla_{i\alpha}\}
\end{aligned}
$$

$$
\{-(\sigma_{\mu\nu})_\alpha{}^\beta[\nabla^\nu, G_{i\beta}] + (\bar{\sigma}_\mu C)^{\dot{\beta}}{}_\alpha[\overline{G}^j{}_{\dot{\beta}}, W_{ij}], \overline{\nabla}^i{}_{\dot{\alpha}}\}\Big)
$$

$$
\left| \quad \left(\begin{aligned}
\{[\nabla^\nu, \overline{G}^i{}_{\dot{\beta}}], \nabla_{i\alpha}\} &= -2i(\sigma^\rho)_{\alpha\dot{\beta}}[\nabla^\nu, g_\rho] - i(\bar{\sigma}^\nu C)^{\dot{\gamma}}{}_\alpha\{\overline{G}_{i\dot{\gamma}}, \overline{G}^i{}_{\dot{\beta}}\}, \\
\{[G^j{}_\beta, W^i{}_j], \nabla_{i\alpha}\} &= -5i\{G_{j\alpha}, G^j{}_\beta\} + \frac{1}{2}C_{\alpha\beta}[[\nabla_z, W_{ij}], W^{ij}], \\
\{[\nabla^\nu, G_{i\beta}], \overline{\nabla}^i{}_{\dot{\alpha}}\} &= 2i(\sigma^\rho)_{\beta\dot{\alpha}}[\nabla^\nu, g_\rho] + i(C\bar{\sigma}^\nu)_{\dot{\alpha}}{}^\gamma\{G^i{}_\gamma, G_{i\beta}\}, \\
\{[\overline{G}^j{}_{\dot{\beta}}, W_{ij}], \overline{\nabla}^i{}_{\dot{\alpha}}\} &= 5i\{\overline{G}_{j\dot{\alpha}}, \overline{G}^j{}_{\dot{\beta}}\} - \frac{1}{2}C_{\dot{\alpha}\dot{\beta}}[[\nabla_z, W^{ij}], W_{ij}],
\end{aligned}\right)\right.
$$

$$
= -3[\nabla^\rho, g_\rho] - 4\{\overline{G}_{i\dot{\gamma}}, \overline{G}^i{}^{\dot{\gamma}}\} - 4\{G_{i\gamma}, G^{i\gamma}\} + i[W^{ij}, [\nabla_z, W_{ij}]],
$$

so that

$$
[\nabla^\mu, g_\mu] = -\{\overline{G}_{i\dot{\gamma}}, \overline{G}^i{}^{\dot{\gamma}}\} - \{G_{i\gamma}, G^{i\gamma}\} + \frac{i}{4}[W^{ij}, [\nabla_z, W_{ij}]]. \tag{C.28}
$$

References

[1] J. Saito, Soryushiron Kenkyu (Kyoto) 111 (2005) 117.

[2] A. D'Adda, I. Kanamori, N. Kawamoto and K. Nagata, Nucl. Phys. **B707** (2005) 100, hep-lat/0406029; Nucl. Phys. Proc. Suppl. **140** (2005) 754, hep-lat/0409092.

[3] A. D'Adda, I. Kanamori, N. Kawamoto and K. Nagata, hep-lat/0507029.

[4] E. Witten, Comm. Math. Phys. **117** (1988) 353.

[5] E. Witten, Comm. Math. Phys. **118** (1988) 411.

[6] L. Baulieu, I. M. Singer, Nucl. Phys. Proc. Suppl. **15B** (1988) 12.

[7] R. Brooks, D. Montano and J. Sonnenschein, Phys. Lett. **B214** (1988) 12.

[8] J. M. F. Labastida and M. Pernici, Phys. Lett. **B212** (1988) 56; Phys. Lett. **B213** (1988) 319.

[9] D. Birmingham, M. Rakowski and G. Thompson, Phys. Lett. **B212** (1988) 187; Phys. Lett. **B214** (1988) 381; Nucl. Phys. **B315** (1989) 577.

[10] M. F. Sohnius, Nucl. Phys. **B136** (1978) 461.

[11] M. F. Sohnius, K. S. Stelle and P. C. West, Phys. Lett. **92B** (1980) 123.

[12] M. F. Sohnius, K. S. Stelle and P. C. West, Nucl. Phys. **B173** (1980) 127.

[13] P. Fayet, Nucl. Phys. **B149** (1979) 137.

[14] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press (1992).

[15] S. J. Gates Jr, M. T. Grisaru, M. Rocěk and W. Siegel, *Superspace, or One thousand and one lessons in supersymmetry*, Addison-Wesley (1983), hep-th/0108200.

[16] M. F. Sohnius, *Introducing Supersymmetry*, Phys. Reports 128 (1985) 39.

[17] S. K. Donaldson, J. Diff. Geom. **18** (1983) 279.

[18] T. Eguchi and S.-K. Yang, Mod. Phys. Lett. **A5** (1990) 1693.

[19] J. M. F. Labastida and P. M. Llatas, Nucl. Phys. **B379** (1992) 220.

[20] M. Alvarez and J. M. F. Labastida, Nucl. Phys. **B437** (1995) 356, hep-th/9404115; Phys. Lett. **B315** (1993) 251, hep-th/9305028.

[21] J. P. Yamron, Phys. Lett. **B213** (1988) 325.

[22] C. Vafa and E. Witten, Nucl. Phys. **B431** (1994) 3, hep-th/9507050.

[23] N. Marcus, Nucl. Phys. **B452** (1995) 331, hep-th/9506002.

[24] M. Blau and G. Thompson, Nucl. Phys. **B492** (1997) 545, hep-th/9612143.

[25] J. M. F. Labastida and C. Lozano, Nucl. Phys. **B502** (1997) 741, hep-th/9709192.

[26] N. Kawamoto and T. Tsukioka, Phys. Rev. **D61** (2000) 105009, hep-th/9905222.

[27] J. Kato, N. Kawamoto and Y. Uchida, Int. J. Mod. Phys. **A19** (2004) 2149, hep-th/0310242.

[28] J. Kato, N. Kawamoto and A. Miyake, Nucl. Phys. **B721** (2005) 229, hep-th/0502119.

[29] J. Kato and A. Miyake, in preparation.

[30] J. Wess and B. Zumino, Nucl. Phys. **B78** (1974) 1.

[31] S. Ferrara and B. Zumino, Nucl. Phys. **B79** (1974) 413.

[32] A. Salam and J. Strathdee, Phys. Lett. **51B** (1974) 353.

[33] P. Fayet, Nucl. Phys. **B113** (1976) 135.

[34] F. Gliozzi, J. Scherk and D. Olive, Nucl. Phys. **B122** (1977) 253.

[35] S. Ferrara, Lett. Nuovo. Cim. 13 (1975) 629.

[36] P. Di Vecchia and S. Ferrara, Nucl. Phys. **B130** (1977) 93.

[37] J. Wess and B. Zumino, Phys. Lett. **66B** (1977) 361.

[38] J. Wess, in *Topics in Quantum Field Theory*, J. A. de Azcárraga, ed., Salamanca (1977).

[39] R. Grimm, M. F. Sohnius and J. Wess, Nucl. Phys. **B133** (1978) 275.

[40] L. Brink, J. H. Schwarz and J. Scherk, Nucl. Phys. **B121** (1977) 77.

[41] B. Milewski, Phys. Lett. **B112** (1982) 148.

[42] P. Claus, B. de Wit, M. Faux, B. Kleijn, R. Siebelink and P. Termonia, Phys. Lett. **B373** (1996) 81, hep-th/9512143.

[43] I. Gaida, Phys. Lett. **B373** (1996) 89, hep-th/9512165.

[44] N. Dragon, E. Ivanov, S. M. Kuzenko, E. Sokatchev and U. Theis, Nucl. Phys. **B538** (1999) 441, hep-th/9805152.

[45] A. Hindawi, B. A. Ovrut and D. Waldram, Phys. Lett. **B392** (1997) 85, hep-th/9609016.

[46] M. F. Sohnius, Nucl. Phys. **B138** (1978) 109.

[47] T. Kugo and P. Townsend, Nucl. Phys. **B221** (1983) 357.

[48] J. Polchinski, *String Theory*, Cambridge University Press (1998).